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Nijholt, E.C.

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CHAPTER 1

INTRODUCTION

1.1 Dynamical Systems and Bifurcations

“If we knew exactly the laws of nature and the situation of the universe at the initial moment, we could predict exactly the situation of the same universe at a succeeding moment.”

This quote, usually attributed to nineteenth century French mathematician Henri Poincaré, illustrates a key concept in all of science; the notion that ‘things can be predicted’ from previous data. Even though quantum mechanics tells us that the situation is at least a bit more complicated than what is suggested in this quote, the underlying idea has not lost its value. A little addendum might be in order though: whereas Poincaré suggests that an exact understanding of the state of the universe and of the laws of this universe should do the trick, I personally think a third ingredient is necessary. This would be a perfect (or at least highly impressive) understanding of how the state of the universe *reacts* to the laws of nature. This is where the mathematical field of *dynamical systems* comes into play.

To illustrate: many people might still remember Newton’s second law of motion from high school. It reads

$$F = m \cdot a. \tag{1.1.1}$$

This is an example of a law of nature. Here F is some force, m is the mass of an object and a is the acceleration of that object. For example, we might

throw a penny off the Eiffel Tower, and wonder about its height after some time. If we know all the data of this situation, such as the height of the Eiffel Tower, the mass of the penny and the force of gravity, then Poincaré's observation tells us that we can solve for the height of the penny after, say, a second of falling. Herein lies the difficulty though, equation (1.1.1) does not tell us anything about the position of the penny. It only gives us the penny's acceleration a . That is, we only know about the change of the velocity, which is 'the change of the change' of the position. Fortunately, it is not hard to solve for the height of the penny in this particular situation, but it illustrates a common problem in physics: the laws of nature describe how things change, but they often do so only obliquely.

The field of mathematics that is concerned with predicting how a system changes over time is called *dynamical systems theory*. Very often, this field of mathematics is called upon to predict how a physical event plays out. Think about a falling object, the motion of the planets, a chemical reaction, oceanic currents or the change in voltage over a computer chip. Other times, the techniques and machinery from dynamical systems theory are used to aid other fundamental fields of mathematics. Whatever the case, questions about dynamical systems can become very hard. As a matter of fact, many successes from this area of mathematics are due not to long and complicated calculations, but rather to finding the right questions to ask (as well as the right questions to avoid).

Be it by chaos, the sheer size of the model or a mathematical quirk of the system, many questions from dynamical systems theory cannot be solved explicitly. There are many examples of relatively simple dynamical systems where we cannot solve for the exact state of the model in the succeeding moment. Often though, we do not need to know the exact state of the model, but rather the approximate, *qualitative* information. That is, we might want to know if a satellite can be brought into orbit, if an animal population remains stable over time or if a decision can be made by a committee, but we are not interested in the exact height of the orbit, the precise number of animals or the exact number of votes in favor. As it turns out, questions like these can often be solved more easily than those asking for precise values. Another reason why the former questions are the right ones to ask, is because they are more *stable* or *robust* in nature. Many things in the world cannot be measured with absolute certainty, but whereas exact outcomes will always vary as initial conditions do, qualitative information often remains unchanged; the satellite will probably still circle the earth if it has a slightly different mass, but the height at which it does will most likely be altered.

Whereas precise values will alter every time a parameter is tweaked, qua-

litative information only changes when the parameter value crosses a specific threshold. This is the idea behind a *bifurcation*. It can roughly be defined as a qualitative change in the dynamics of a parameter-dependent system, caused by a slight change in that parameter. As an example, consider the most basic mathematical model of a dynamical system: an ordinary differential equation, or ODE for short, is an equation of the form

$$\frac{dx(t)}{dt} = f(x). \quad (1.1.2)$$

Here f is a *vector field* on some space V , meaning that it assigns a speed and direction of movement to each point in V . The function $x(t)$ is a one-dimensional curve in V , parameterised by t in some interval in \mathbb{R} (very often t is, of course, just time). Equation (1.1.2) simply tells us that the curve $x(t)$ changes in a direction and with a magnitude dictated by the vector field f . It is our job to find $x(t)$ (or to at least say something meaningful about it) when f is given. Note that we have already seen this set-up, namely in the situation of the falling penny. There, we wanted to find the height of the penny after a second of falling, so $x(t)$ for $t = 1$. However, we were only given its acceleration

$$a := \frac{d^2x(t)}{dt^2} \quad (1.1.3)$$

by equation (1.1.1). Even though the equation

$$\frac{d^2x(t)}{dt^2} = \frac{F}{m} \quad (1.1.4)$$

that we get from rewriting equation (1.1.1) involves a second derivative instead of a first, one can easily recast this equation to one of the form (1.1.2). Returning to equation (1.1.2), it could be that the vector field f depends on some parameter that is not completely known, or that might slowly shift over time. We therefore get a *family of ODE's*, denoted by

$$\frac{dx(t)}{dt} = f(x, \lambda). \quad (1.1.5)$$

What we have added here is the parameter λ , which takes values in some parameter space $\Omega \subset \mathbb{R}^k$. Note that λ does not change over time in this model. Hence, for fixed values of λ we get equations of the form (1.1.2) back for different vector fields f . Of course we expect these vector fields to vary smoothly in λ , but this can still cause abrupt changes in the qualitative behaviour of the system.

For example, if we set $f(x, \lambda) = x^2 - \lambda$, with both x and λ taking values in \mathbb{R} , then $f(x, \lambda)$ is always positive if λ is negative. As a result, for these values of λ any initial state will just grow forever as time progresses. If λ is positive though, the vector field f has zeroes at $x = \sqrt{\lambda}$ and at $x = -\sqrt{\lambda}$. This means that the dynamics does not change if it starts at these points, and we call them *steady state points*. Hence, this is an example of a *steady state bifurcation*. What is more, it can be shown that the point $x = -\sqrt{\lambda}$ is *attractive*, meaning that it has a neighbourhood of points that will approach it as time progresses. Note that this situation is very different from what we saw for $\lambda < 0$. Figure 1.1 shows a sketch of the situation. This particular steady state bifurcation is known as a *saddle-node bifurcation*. Steady state bifurcations are by no means the only bifurcations possible. For example, there are scenarios where periodic orbits appear, most notably in a *Hopf bifurcation*. Likewise, saddle-node bifurcations are not the only steady state bifurcations. However, they are in some way the most ‘likely’ ones, as we will see below.

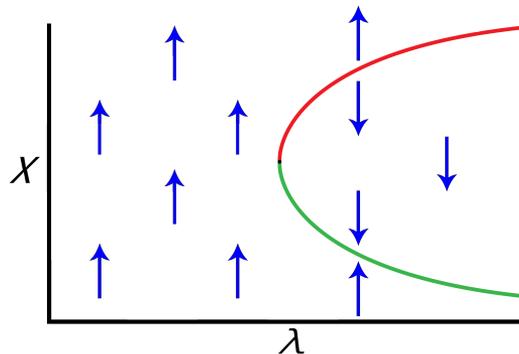


Figure 1.1: A saddle-node bifurcation. Depicted here are the steady state points. That is, points (x, λ) where $f(x, \lambda) = 0$. Green denotes stable steady state points, whereas red denotes unstable steady state points. The blue arrows show the direction of the vector field.

1.2 Network Structures

Many dynamical systems modeling real-life phenomena come with an inherent network structure. Think about the brain, social media, the spread of a disease, social decision making or competing populations of animals. In all cases, there are clearly distinguished units or *cells* together with a

structure of dependence between them. Mathematically, this means that the cells correspond to different variables. These variables then come with a prescribed set of rules determining what other variables they influence and what kind of influence this is.

For example, consider three populations of animals that live in the same area. Two of them, populations A and B , might belong to the same species, but form different competing colonies. They might be predators, and hunt for species C . Hence, the populations of A and B both rely on that of C , and they do so in the same manner. Likewise, the population of C depends on that of A and B . Of course, the population of species C also depends on itself, but it does so in a very different way from how it depends on the population of its predators. If only, this is because a larger population of species C will cause a greater increase in that same population (neglecting effects of overpopulation), whereas a greater population of predators A and B will generally cause a lower increase in population C . On top of this, A and B might also compete for territory, and hence negatively influence each other's population. Putting all this together, we get the ODE given by

$$\begin{aligned}\dot{x}_A &= f(x_A, x_C, x_B) \\ \dot{x}_B &= f(x_B, x_C, x_A) \\ \dot{x}_C &= g(x_C, \overline{x_A}, \overline{x_B}),\end{aligned}\tag{1.2.1}$$

where \dot{x}_I , $I \in \{A, B, C\}$, is just shorthand notation for the derivative with respect to time. That is,

$$\dot{x}_A := \frac{dx_A(t)}{dt}\tag{1.2.2}$$

and so on.

Now, populations A and B correspond to the same species. Therefore, we assume that their variables x_A and x_B take values in some same *phase space* V . For this same reason, they react to their environment by the same *response function* f . In this function, the first slot describes how each population depends on its own population ‘moments ago’. The second slot (the red one) captures the effect that the population of prey animals has and the third slot (the blue one) describes the effect of territorial competition. Population C represents a different species and therefore corresponds to a variable x_C in some (a priori) different phase space W . Likewise, it has a different response function g . Again, the first slot of g corresponds to the effect of the population on the growth of itself. (This variable is depicted in grey rather than in black to emphasise the convention that different types of variables can, by definition, never receive input of the same kind). The

green variables signify the effects of A and B on the population of C . The bar over these variables means that we may freely interchange them without effect. That is, we have

$$g(x_C, x_A, x_B) = g(x_C, x_B, x_A), \quad (1.2.3)$$

for all values of x_A , x_B and x_C . This is done because the effects of A and B on C should be identical. Except for the response functions f and g and the phase spaces V and W , all of this information is included in the graph of Figure 1.2. Here the cells depict the different populations A , B and C and the arrows correspond to the different effects they have on each other. (We have left out the black and grey self loops for convenience).

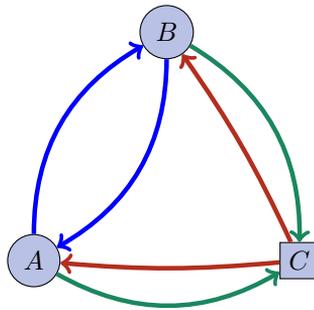


Figure 1.2: A network describing the relations between different populations of animals A , B and C .

In this thesis we will mainly look at *homogenous* networks, meaning that all cells are of the same type and all response functions are therefore identical. The reason for this is mainly convenience; many of the results still hold for inhomogeneous networks, but become more technical and notationally much less transparent. As all cells are identical, they each receive input from the same number of arrows for each color of arrow. What we will also assume is that this number is always 1. That is, each cell receives input from exactly one arrow of each color. This condition is more important though, as many of the techniques in this thesis have not yet been extended to the general case in a satisfactory way.

1.3 Generality

The main purpose of this thesis is understanding *generic* phenomena, most notably generic bifurcations, in network dynamical systems. To see what

this term means, we revisit the family of ODEs in equation (1.1.5):

$$\dot{x} = f(x, \lambda). \tag{1.3.1}$$

Let us assume that both x and λ take values in \mathbb{R} . If we consider equation (1.3.1) for all possible smooth vector fields $f(x, \lambda)$, then it seems that any steady state bifurcation could occur. In fact, it is known that any closed subset of \mathbb{R}^2 can occur as the zero-set of some smooth map $f : \mathbb{R}^2 \rightarrow \mathbb{R}$. However, in a way some bifurcations are more ‘likely’ than others. To illustrate, let us assume that f has a steady state point at $(x, \lambda) = (0, 0)$. In other words, it holds that $f(0, 0) = 0$. Now, if the derivative of f in the x -direction does not vanish at $(0, 0)$, then the *implicit function theorem* tells us that there is no (interesting) bifurcation near $(0, 0)$. More precisely, it follows that the steady state points are locally parametrised by λ . Therefore, we also set this derivative equal to zero. Equation (1.3.1) now becomes

$$\dot{x} = a\lambda + bx^2 + \text{‘higher order terms’}, \tag{1.3.2}$$

for a and b in \mathbb{R} . As it turns out, as long as both a and b are unequal to 0, there always occurs a saddle-node bifurcation. As a general function f will not have any of these two coefficients vanish, we say that a saddle-node bifurcation is the *generic* steady state bifurcation for systems of the form (1.3.1).

As it turns out, this result relies strongly on the class of functions f that we allow. For example, if we only allow for smooth maps f satisfying the symmetry condition $f(-x, \lambda) = -f(x, \lambda)$ for all values of x and λ in \mathbb{R} , then we generically get a *pitchfork* bifurcation instead of a saddle-node bifurcation. See Figure 1.3. This seems paradoxical: we know that for a general vector field we almost always get a saddle-node bifurcation, shouldn’t we then also get this bifurcation for most symmetric vector fields? The answer is in the numbers a and b of equation (1.3.2). These are non-zero for most vector fields f , but this is not true for most symmetric vector fields f . As a matter of fact, the condition $f(-x, \lambda) = -f(x, \lambda)$ forces b to be 0. Another way to see that the saddle-node bifurcation is not the generic situation here is by noting that we have $f(0, \lambda) = f(-0, \lambda) = -f(0, \lambda)$. Therefore, we have $f(0, \lambda) = 0$ for all values of λ . This completely excludes the saddle-node bifurcation! We see that symmetry can apparently force behaviour that is otherwise unheard of.

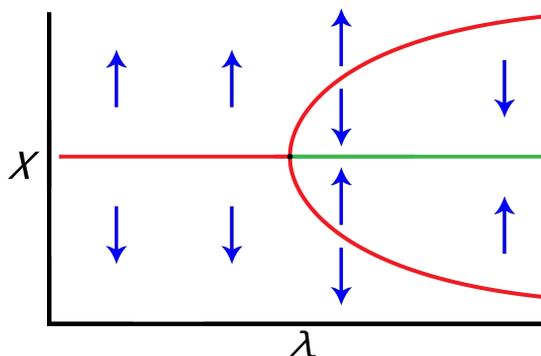


Figure 1.3: An example of a pitchfork bifurcation.

1.4 Generic Bifurcations in Network Dynamical Systems

As it turns out, many network vector fields likewise show unusual behaviour as a generic occurrence. For example, we could look at all equations of the form

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2, x_2, x_3, \lambda) \\ \dot{x}_2 &= f(x_2, x_3, x_2, x_3, \lambda) \\ \dot{x}_3 &= f(x_3, x_3, x_2, x_3, \lambda), \end{aligned} \tag{1.4.1}$$

for some response function f . These correspond to the network of Figure 1.4. There are two bifurcation scenarios that generically occur for this class of network vector fields. The first of these is the familiar saddle-node bifurcation. The second one is more surprising and is shown in Figure 1.5. Needless to say this scenario is highly unusual in a general class of vector fields (without a network structure).

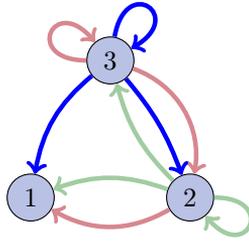


Figure 1.4: A homogeneous network with three cells.

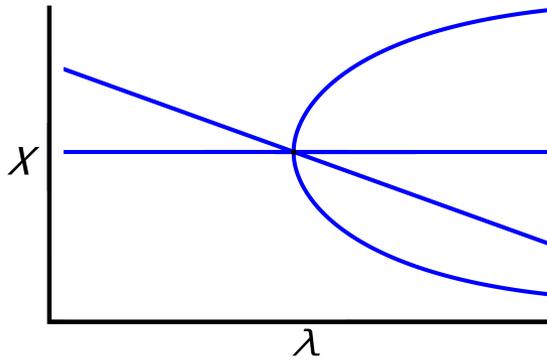


Figure 1.5: A very unusual steady state bifurcation.

This thesis aims at explaining why network dynamical systems display such anomalous behaviour. We have already hinted at a possible answer: another feature that forces unusual phenomena is symmetry. As it turns out, a large class of network vector fields indeed admits a description in terms of symmetry. It is often not as visible as in the equation $f(-x, \lambda) = -f(x, \lambda)$ though, but instead lies ‘hidden’ in some extension of the original system. Furthermore, the kind of symmetry that rears its head in network dynamical systems often fails to form a group, as is classically the case. Rather, it represents some weaker notion, such as a so-called semigroup or monoid. Nevertheless, many techniques from the study of group-symmetry can still be applied or adapted to this new setting. This idea of using *hidden symmetry* runs like a common thread through the entire thesis.

1.5 Structure of the Thesis

The main body of this thesis consists of four chapters, each of which is a self-contained scientific article. The first of these, chapter 2, is contained in the article “Graph fibrations and symmetries of network dynamics”, (*J. Differ. Equations* 261, 2016). It introduces the language of *graph fibrations*, used to visualise relations between networks graphs. Using this formalism, it is shown that a large class of network vector fields has some form of hidden symmetry. This means that the networks themselves might not have any symmetry, but instead form the restriction of a very symmetric system called the *fundamental network*. The article “Graph fibrations and symmetries of network dynamics” contains some additional results using graph fibrations that are not included in this thesis.

Chapter 3 has been published as “Center Manifolds of Coupled Cell Networks”, (*SIAM Journal on Mathematical Analysis* 49.5, 2017). In here, the machinery of *center manifold reduction* is adapted to fit the setting of network dynamical systems. In particular, it is shown that the resulting *reduced vector fields*, in a way, still retain their original network structure. Moreover, this reduction respects *synchrony*, a very important network property whereby different cells behave in unison. Ultimately, these results rely on the formalism introduced in the previous chapter. This is because network structures are notoriously hard to track through many mathematical constructions, whereas (hidden) symmetry seems to be significantly better behaved.

With the tools of chapter 3 at hand, chapter 4 focusses on proving some first results relating generic bifurcations in network systems to their graphs. More precisely, it is shown that a special structure in a graph, called a *projection block*, may be shrunk to a single cell to obtain a smaller graph. Many of the dynamical properties of this smaller network then still hold true for the original one. Ultimately, the goal is to use techniques such as these to fully decompose any network into smaller, more manageable components to understand its dynamical behaviour. Chapter 4 corresponds to the article “Projection blocks in homogeneous coupled cell networks”, (*Dynamical Systems* 32.1, 2017).

Finally, chapter 5 is available as a preprint on arxiv.org, under the name “Transversality in Dynamical Systems with Generalized Symmetry”. It focusses on adapting a well-known technical result on group symmetry to the more exotic kinds of symmetry that turn up in the study of networks. In broad terms, it describes what linear behaviour to consider when predicting bifurcations in systems with such a symmetry.