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SEARCH AND MOBILITY IN A HOUSING MARKET

WITH LIMITED SUPPLY

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Search and Mobility in a Housing Market with Limited Supply

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1. Introduction

Housing markets in many European cities are characterized by strong government intervention in the form of price controls, subsidies, zoning etc. This is one of the reasons why frequently disequilibria are observed on the urban housing market as a whole and its various submarkets; these disequilibria manifest themselves in queues of households waiting and searching for better dwellings on the one hand, and vacancies on the other. In order to be able to describe these phenomena models of housing markets need to be developed that take these disequilibria into account explicitly. In the present paper some steps in that direction will be taken.

Housing market research during the last decade his been focussed in residential location models using discrete choice theory and random utility models as a point of departure (see e.g. Clark and Van Lierop, 1986). This has given rise to a strong micro bias and also a demand side orientation to housing market research. In the present paper an effort is made to study urban housing markets from a meso perspective. An integral picture of a dynamic market where households enter, change in size and finally disappear will be given. Housing supply is heterogeneous and families of different size and social class will have different preferences. The model accordingly describes the housing market 'career' of a household during its life-cycle.

In the model vacancy chains are taken into account since each moving household leaves a vacant dwelling that may be occupied by another household in the next period. The usual way of dealing with vacancy chains is by means of Markov models (see Porell, 1982). Our model is not a usual Markov model however, since it follows from the specification that the transition probabilities are not constant. These probabilities depend on the stock variables in the housing market, so that a non-linear dynamic system is obtained. Consequently, existence, uniqueness and stability of an equilibrium solution are no longer self evident (see Haag and Weidlich, 1984).

The present model bears some resemblance to a dynamic stock-flow model developed by Weibull (1980). Like Weibull we deal with the housing market as a system of interrelated submarkets that usually will be in a state of permanent disequilibrium (i.e. no Walrasian equilibrium will appear in a stationary state). The present model however gives a more detailed picture of the dynamics of the households. Anas and Cho (1986) also deal with a (partially) regulated housing market but do not deal explicitly with household dynamics. In the present model prices do not play an explicit role. The market is assumed to be fully regulated: dwellings are rented and rents are assumed to be fixed.

Section 2 of the paper is devoted to the presentation of a simple version of the model in which N types of dwellings are distinguished and houselholds are assumed to be homogeneous. Existence, uniqueness and stability of the equilibrium solution will be discussed. In section 3 this model will be extended by relaxing the assumption of homogeneous households. Different sizes of households will be distinguished and life-cycle phenomena are taken into account. In this extended model households may move, either because they found a dwelling which is more attractive than the present one (their class being unchanged) or because they recently entered another class so that their preferences and possibilities have changed. Section 4 is devoted to an illustration by means of numerical simulation experiments. Section 5 contains some concluding remarks.

2. Basic Model

This section is devoted to the presentation of a simplified model in which there is only one type of household, while there exist N types of dwellings. All households have the same preference-ordering with respect to these types of dwellings. If they are not able to occupy a dwelling of the most preferred type, they will for some time accept a less preferred one and move from there to higher preferred types. As long as they do not occupy a dwelling of the most preferred type they continue to search.

Total population is assumed constant, but in each period some households disappear, while new ones are formed. Starting households initially do not occupy a dwelling; vacancies occur because of disappearing households. In the models that will be discussed in this paper starting households include both newly formed households and immigrating ones. Similarly, disappearing households include terminating as well as outmigrating households.

2.1. Some definitions and identities

There are N types of dwellings. Type 1 is most preferred, type N is least preferred. The number of dwellings of each type is denoted as W_n (N=1,...,N). The total number of dwellings is W:

$$W = \sum_{n=1}^{N} W_n \qquad W_n > 0 \text{ for } n=1,\dots,N \tag{2.1}$$

The number of households occupying a dwelling of type n is denoted as B_n (n= 1,...,N). The number of households that do not yet occupy a dwelling is B_{N+1}. The total number of households is B:

$$B = \sum_{n=1}^{N+1} B_n \qquad 0 \leq B_n \leq W_n \text{ for } n=1,\dots,N \tag{2.2}$$

The number of vacant dwellings of type n is L_n:

$$L_n = W_n - B_n \qquad n=1,\ldots,N \tag{2.3}$$

It follows that:

$$L = B_{N+1} + W - B \tag{2.4}$$

From this equation it may be inferred that the total number of vacant dwellings consists of two parts: one caused by overall excess supply (equal to W-B if W>B and equal to zero otherwise) and one caused by friction (equal to B_{N+1} if W>B and to B_{N+1} + W-B otherwise).

All households that do not occupy a dwelling of the most preferred type are searching, so that the total number of searching households equals:

$$S = B - B_1 \tag{2.5}$$

These households have, in one way or another, to be confronted with the vacant dwellings.

2.2. The distribution mechanism

A simple distribution mechanism is the following one: draw randomly and without replacement a vacant dwelling from the total stock of vacant dwellings and a searching household from the total stock of searching households and confront them with each other; continue this procedure until there are no vacant dwellings or no searching households left over.

A household that received an offer may accept or refuse. In the former case allocation takes place and the household occupies the vacant dwelling. A new vacancy will emerge if this household did already occupy a dwelling; if it did not yet occupy a dwelling this will not be the case. If the offer is not accepted the household goes on searching. If the offer is accepted search will only be stopped if the offer concerned a dwelling of type 1, otherwise it will be continued immediately.

It is assumed that every searching household accepts every offer that concerns a dwelling of a type preferred to the one presently occupied. One may interpret this as an assumption of zero moving costs. Households that did not yet occupy a dwelling are assumed to accept all offers.

The probability q that a searching household receives an offer may on the basis of the foregoing be determined as:

$$\left.\begin{array}{ll} q = L/S & \text{if } L < S \\ q = 1 & \text{if } L \geq S \end{array}\right\} \tag{2.6}$$

Given that an offer is received, the probability that it will be accepted depends on the type of dwelling presently occupied by the household that receives the offer and on the type of dwelling that the offer is concerned with. The fraction of vacant dwellings that are of type n is π_n:

$$\pi_n = L_n/L \qquad n = 1,\dots,N \qquad (2.7)$$

The probability that a household, presently occupying a dwelling of type n accepts an offer is $\sum_{i=1}^{n-1} \pi_i$ while that probability for a household not occupying a dwelling at present is equal to 1 (= $\sum_{i=1}^{N} \pi_i$).

The probability that a vacant dwelling will be offered to a searching household is:

$$\left.\begin{array}{ll} r = S/L & \text{if } S < L \\ r = 1 & \text{if } S \geq L \end{array}\right\} \qquad (2.8)$$

The probability that a vacant dwelling, given that it is offered to a searching household, will be accepted depends on its type and the type of dwelling presently occupied by the household to which it is offered. The fraction of searchers presently occupying a dwelling of type n (n = 2,...,N) or no dwelling at all is ρ_n.

$$\rho_n = B_n/S \qquad n = 2,\dots,N+1 \qquad (2.9)$$

The probability that an offered dwelling of type n will be accepted is thus equal to $\sum_{j=n+1}^{N+1} \rho_j$. Offers of dwellings of type 1 are always accepted since $\sum_{j=2}^{N+1} \rho_j = 1$.

2.3. Streams of removers

The total number of households moving from a dwelling of type n into one of type m, s_{nm} is equal to:

$$s^t_{nm} = (1-v)\, q^t\, \pi^t_m\, B^{t-1}_n \qquad m = 1,\dots,n-1;\; n = 1,\dots,N+1 \qquad (2.10)$$

The stream of removers from type n to type m dwellings is thus equal to the total number of households presently occupying a dwelling of type n times the probability q that an offer will be received times the probability that a received after concerns a dwelling of type m, taking into account the probability of household termination v.

If q is smaller than 1 it follows from (2.6) and (2.5) that:[1]

$$s_{nm}^{t} = (1-v)\ (W_m - B_m^{t-1})\ B_n^{t-1}/(B - B_1^{t-1}) \qquad (2.11)$$

Taking into account the disappearance of households (at a rate v, 0<v<1) the following difference equations will hold:

$$\left.\begin{aligned} B_1^t &= (1-v)\ B_1^{t-1} + \sum_{m=2}^{N+1} s_{m1}^t \\ B_n^t &= (1-v)\ B_n^{t-1} - \sum_{m'=1}^{n-1} s_{nm'}^t + \sum_{m=n+1}^{N+1} s_{mn}^t \quad n=2,\dots,N \\ B_{N+1}^t &= (1-v)\ B_{N+1}^{t-1} - \sum_{m'=1}^{N} s_{N+1m'}^t + vB \end{aligned}\right\} \quad (2.12)$$

The last one of these equations expresses the assumption that all starting households initially do not occupy a dwelling.

Substitution of (2.10) into (2.11) and writing $B - \sum_{m=1}^{N} B_m$ for $\sum_{m'=n+1}^{N+1} B_m$ yields:

$$\begin{aligned} B_1^t &= (1-v)\ \{\ B_1^{t-1} + q^t\ \pi_1^t\ (B - B_1^{t-1})\} \\ B_n^t &= (1-v)\ \{(1- q^t \sum_{m'=1}^{n-1} \pi_{m'}^t)\ B_n^{t-1} + q^t\ \pi_n^t(B - \sum_{m=1}^{n} B_m^{t-1})\} \quad n=2,\dots,N \\ B_{N+1}^t &= (1-v)\ \{(1 - q^t)\ B_{N+1}^{t-1}\} + vB \end{aligned} \qquad (2.13)$$

It should be noted that in the present model the values of q^t and the π_m^ts are not fixed exogenously, but will be endogenously determined (cf. equations 2.6 and 2.7). Consequently, this model is not an ordinary Markov model with fixed parameters. The transition probabilities depend on the state of the system itself. Nevertheless the resemblance of the system of equations (14) to a Markov model is of some use as will become clear in the next section. The dependence of the transition probabilities on the state of the system means essentially that supply constraints for housing are taken into account. Overcrowding in certain dwelling types is avoided in this model.

The purpose of the model is to explain the distribution of the population over the different types of dwellings, given the numbers of dwellings of each type, the preferences of the households, the distribution mechanism and the stopping/starting rate.

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1) In this subsection an upper index t is used for the endogenous variables to denote their time dependence. Later on this index will sometimes be suppressed in order to avoid unnecessary complexity of notation.

2.4 Stationary states

The system of equations (2.13) may be used to explore the dynamics of the present model. This system is in a dynamic equilibrium if the values of the B_n's remain the same as time moves on. Such an equilibrium is called a stationary state and one may wonder whether such a state exists in the present model.

An affirmative answer can be given to this question. Two methods to reach this answer will be mentioned. The first consists of the use of a so-called fixed-point-theorem. In this way it can be shown that the system (2.13) has a steady state solution under very general conditions. The fixed point theorem does not tell us however what values of the B_n's will be associated with the stationary state. Therefore in this section another method will be elaborated, viz. direct computation of the stationary state. Since this method is less general than the one that uses a fixed-point-theorem it will still be worthwile to pay some attention to the latter. This is done in Appendix A.

The stationary state of the system (2.13) will now be computed for the case in which $L \leq S$, that may be associated with excess-demand on the housing market. This seems to be the most realistic case also. (In the model this requirement will always be fullfilled if $W \leq B$ since then $L \leq B_{N+1} \leq B - B_1 = S$. This condition is a sufficient but not a necessary one.) If $L \leq S$ q is smaller than 1 and $q\pi_m$ may be written as:

$$q\pi_m = (W_m - B_m)/(B-B_1) \qquad m=1,\dots,N \qquad (2.14)$$

If this is substituted in equations (2.13) it becomes clear that this system is a recursive one: the values of all B_n's are determined by those of $B_1,\dots, B_n$ one period before and there is no influence of these of $B_{n+1},\dots,B_{N+1}$. In a stationary state all B_n's have the same value in successive periods. Imposing this condition it follows for the first equation of (14) (after substitution of 15):

$$B_1 = (1-v)W_1 \qquad (2.15)$$

For $n \geq 2$ a more complicated equation will be found after substitution of (15):

$$(B-B_1)B_n = (1-v)\{[(B-B_1)-\sum_{m=1}^{n-1}(W_m - B_m)]B_n + (W_n - B_n)(B-\sum_{m=1}^{n-1}B_m - B_n)\}$$

$$n=2,\dots,N \qquad (2.16)$$

This equation is quadratic in B_n and may be solved for that variable. The result is (see Appendix B):

$$B_n = 1/2\ \{c_{n1} - \sqrt{c_{n1}^2 - 4c_{n2}}\} \qquad n=2,\dots,N \qquad (2.17)$$

where:

$$c_{n1} = [\frac{v}{1-v}(B - B_1) + \sum_{m=1}^{n-1}(W_m - B_m) + (B - \sum_{m=1}^{n-1} B_m) + W_n]$$

$$c_{n2} = W_n(B - \sum_{m=1}^{n-1} B_m)$$

Finally the value of B_{N+1} in the stationary state can be obtained as $B - \sum_{n=1}^{N} B_N$ or, alternatively from writing out the last equation of (14), the following formula can be derived:

$$B_{N+1} = vB/\{1 - (1-v)\ (1 - \frac{\sum_{m=1}^{N}(W_m - B_m)}{B - B_1}\} \qquad (2.18)$$

Equations (2.15), (2.17) en (2.18) show the possibility to compute the stationary state associated with a set of exogenous variables $W_1,\ldots,W_N$, B and v as long as $L \leq S$, which is guaranteed if $W \leq B$. Since for every n B_n was shown to have a unique value the stationary state will be unique.

Once the existence of a stationary state is established, the question arises whether it will be stable. Since global stability of dynamical systems is usually difficult to prove, attention is often concentrated on local stability. A system is locally stable if it returns to the stationary state after a small (exogenously caused) jump out of such a state.

The dynamical system given by equation (2.13) is locally stable if the eigenvalues of the matrix of its first-order partial derivatives are in absolute value smaller than 1 (see e.g., Hirsch and Smale, 1974, pp. 280-1). Since that system was shown to be a recursive one, this matrix is triangular, and the eigenvalues are equal to its diagonal elements $\partial B_n^t/\partial B_n^{t-1}$:

$$\left.\begin{array}{ll} \partial B_1^t/\partial B_1^{t-1} = (1-v)\ (1 - q\pi_1) & \\ \partial B_n^t/\partial B_n^{t-1} = (1-v)\ [(1-q \sum_{m=1}^{n-1} \pi_m) - q\pi_n] & n = 2,\ldots,N \\ \partial B_{N+1}^t/\partial B_{N+1}^{t-1} = (1-v)(1-q) & \end{array}\right\} \qquad (2.19)$$

From equations (2.19) it will be clear that $|\partial B_n^t/\partial B_n^{t-1}| < 1$ for all $n=1,\ldots,N+1$. So it may be concluded that the stationary states associated with the present model will always be locally stable.

An argument for global stability in the case where $L \leq S$ is provided by the following reasoning. Start by observing that the value of B_1 will after one period always be equal to its stationary state value. If B_2 is not equal to its stationary state value it can be higher or lower. In the former case it can be observed from (2.11) that the stream of removers from type 2 to type 1 dwelling will be greater than in the stationary state (since B_2 is greater than in the stationary state while W_1-B_1 and $B-B_1$ have their stationary state values),

while the total stream of removers into type 2 dwellings will be smaller than in the stationary state (since both $W_2 - B_2$ and $\sum_{n=3}^{N+1} B_N$ will be smaller than in the stationary state).
In the latter case the stream from 2 to 1 will be smaller than in the stationary state, while the total stream into type 2 dwellings will be greater. It may be concluded that B_2 moves in the direction of its stationary state value. As B_2 approaches that value more and more closely the same reasoning leads to the conclusion that B_3 moves in the direction of its steady state value. The same will be true for other B_n's. Although this reasoning gives no formal proof of the global stability of the present system (when L≤S) it strongly suggests this property.

An example of the computation of the stationary state values of B_n and of the movement towards equilibrium is contained in Appendix C.

2.5 Features of the stationary state

	W_1	W_2	W_3		W_N
B_1	+	0	0	.	0
B_2	−	+	0	.	0
B_3	−	−	+	.	.
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	0
B_N	−	.	.	.	+
B_{N+1}	−	−	−	.	−

Table 1. Signs of partial derivatives $\frac{\partial B_n}{\partial W_m}$.

In Table 1 it is shown how the distribution of households among dwelling types is influenced by the size of the dwelling stock for each type. The table presents the signs of the partial derivatives of the number of households living in dwelling type n with respect to the dwelling stock of type m for all m,n. It has been assumed here and in the rest of this section that L≤S. The results indicate that an extension of the housing stock of type m only influences the number of households living in type m or in a lower quality dwelling. Not surprisingly, $\frac{\partial B_n}{\partial W_n}$ is positive for all n, and all remaining partial derivatives are negative. Note that since $B_1+B_2+\ldots+B_{N+1}=B$ (which is constant in the model), the vertical sum of the partial derivatives in Table 1 is zero for all columns. The general conclusion which may be drawn from Table 1, is that the most far reaching effects on the population distribution are generated by the construction of dwellings of the highest quality.

A similar result is obtained when vacancy chains are studied. A vacancy generated in the lowest quality of housing type (N) generates only one moving household (i.e. from position N+1). Similarly, a vacant dwelling of type N-1 may give rise to one moving household if this one comes from position N+1, or a chain consisting of a household moving from N to N-1 leaving a vacant dwelling which is occupied by a household coming from position N-1. The probability of the first possibility is $B_N/(B_N+B_{N+1})$, the probability of the second possibility being $B_{N+1}/(B_N+B_{N+1})$. Note that the latter statement depends on the assumption made about the acceptance behaviour of searching behaviour of searching households, i.e., all households accept an offered dwelling, irrespective of the present type of dwelling occupied, if the offered dwelling is in a higher quality class than the present dwelling. The statement also depends on the way the distribution mechanism is modelled. As indicated in section 2.2 the distribution takes place in a random way, which implies the absence of discriminative policies for starting households, both in a positive or negative way. These two assumptions are reflected by (2.13) where one finds that the transition probabilities of households from all types m>n to a dwelling of type n are equal.

Given the above discussion it can easily be seen that the average number of moves M_n caused by a vacancy of type n is equal to:

$$M_N = 1$$

$$M_{N-1} = [(1)\ B_{N+1} + (1+M_N)\ B_N]/(B_{N+1} + B_N) \qquad (2.20)$$

$$M_n = [(1)\ B_{N+1} + (1+M_N)\ B_N + \ldots\ldots + (1+M_{n+1})\ B_{n+1}]/$$
$$(B_{N+1} + B_N + \ldots\ldots + B_{n+1}) \qquad (n=1,\ldots,\ N-1)$$

These formulas can easily be rewritten as:

$$M_N = 1$$

$$M_n = M_{n+1} + B_{n+1}/(B_{n+1} + \ldots\ldots + B_N + B_{N+1}) \quad (n=1,\ldots,\ N-1) \quad (2.20')$$

so that it is immediately clear that vacancy chains are longer the higher the quality class in which the first vacancy is generated.

We now turn to a discussion of the duration of vacancy and of residence in various dwelling types. The average duration of vacancy for dwelling type n (α_n) is in the stationary situation equal to the ratio of the stock of vacant dwellings in n and the number of dwellings which become occupied per period (I_n):

$$\alpha_n = (W_n - B_n)/I_n \qquad (2.21)$$

From (2.13) we know that

$$I_n = (1-v)\, q\pi_n(B-B_1-\ldots-B_n) \qquad (2.22)$$

Since $q\pi_n=(W_n-B_n)/(B-B_1)$ as already shown in (2.11), one obtains for α_n:

$$\alpha_n=(B-B_1)/\{(B-B_1-\ldots-B_n).(1-v)\} \qquad (2.23)$$

so that the conclusion reads that the duration of vacancy is shortest for n=1 and longest for n=N.

For the average <u>duration of residence</u> a similar result can be obtained. In the stationary state, the duration of residence in n (β_n) is equal to the ratio of the total number of households in n and the number of households leaving n (E_n):

$$\beta_n= B_n/E_n \qquad (2.24)$$

From (2.13), it follows that:

$$E_n= \{v+(1-v)q(\pi_1+\ldots+\pi_{n-1})\}B_n \qquad (2.25)$$

Note that in the stationary state $E_n=I_n$. In (2.24) we use E_n rather than I_n since it is easier to reach conclusions on the dependence of β_n on n. After substituting (2.25) into (2.24) one can derive that the duration of residence is longest for n=1 and shortest for n=N. Thus, duration of residence and duration of vacancy are inversely related in this model.

In the context of this model, generally speaking duration of residence is equal to <u>duration of search</u> before a certain move is made. The only exceptions occur for n=1, where households live without searching and for n=N+1 where households search without having a dwelling.

These outcomes on the duration of vacancy and residence can also be used to derive results on the <u>vacancy rate</u>, which is defined as:

$$\gamma_n=(W_n-B_n)/W_n \qquad (2.26)$$

Since $E_n=I_n$ in the stationary state, it follows from (2.21), (2.24) and (2.26) that:

$$\gamma_n= \alpha_n/(\alpha_n+\beta_n) = 1/(1+\beta_n/\alpha_n) \qquad (2.27)$$

As the duration of vacancy α increases with n, and the duration of residence β_n decreases with n, it may be concluded that the vacancy rate increases with n. It is highest in the class of low quality dwellings.

The various results in this section on vacancy chains, duration of vacancy, etc. all point into the same direction: pressure on the housing market is highest for high quality dwellings. The most effective way to reduce pressure is the construction of high quality dwellings. These conclusions of course strongly depend on the rather simplistic assumptions underlying the model, especially the assumed homogeneity of households. Therefore, an extension of the present model will be given in the next section to take into account the heterogeneity of households.

3. Extensions of the basic model

The model descibed in the previous section can be extended in several ways. In the present section some of these extensions will be dealt with. We will no longer stick to the assumption of identical households; various household types will be distinguished. Consumer behaviour will be discussed. Further, attention will be paid to social mobility and its consequences for residential mobility. Attention will be concentrated on the equations for the streams of removers, since it will in general not be easy to compute steady states in an analytical way.

3.1. Different types of households

A first extension of the model concerns the possibility that different households will have different tastes for the types of dwellings. E.g. the number of persons in a household and its income are highly relevant variables of housing demand. So it will be useful to look at the changes that have to be made in the specification of the model of section 2 to allow for the possibility of different types of households with different tastes.

Different types of households may be distinguished on the basis of their characteristics and the value of their income. It will be assumed that the former can be described by means of a vector $\underline{x}$, while the latter will be denoted as y. Utility depends on the characteristics of the households, on the quantities of the goods consumed, denoted by means of a vector $\underline{q}$, and on the characteristics of the dwelling that is occupied, denoted as a vector $\underline{z}$. The vector $\underline{z}$ can take on only N different values, each associated with one type of dwelling.

$$U = U(\underline{x}, \underline{q}, \underline{z}) \quad (3.1)$$

The prices of all goods are assumed to be given and will be denoted as a vector $\underline{p}$; the price of the n-th type of dwelling is denoted as p^n. If a household would live in a residence of type n, utility is maximized under the budget constraint:

$$\underline{p}'\,\underline{q} + p^n = y \quad (3.2)$$

whereas

$$\underline{z} = \underline{z}^n \quad (3.3)$$

In the last equation $\underline{z}^n$ is the vector of dwelling characteristics associated with the n-th type of dwelling. The maximum level of utility that a household with characteristics $\underline{x}$ at given prices $\underline{p}$ and a given income y is able to realize, given that it occupies a dwelling of type n, can be obtained by substituting the optimal values of $\underline{q}$ in (3.3). This value is denoted U^n:

$$U^n = U^n(\underline{x}, \underline{p}, p^n, y, \underline{z}^n) \quad (3.4)$$

From (3.4) it may be concluded that when all prices are given, the values U^n for the different types of dwellings are determined by the values of $\underline{x}$ and y. It is assumed that on the basis of these variables a finite number of I types of households can be distinguished.

The preference-ordering of the i-th type of household (i=1,...,I) is determined by the values of U^n. In order to formulate the equations for the streams of removers for the model extended to different kinds of households some further notation is introduced. $S_n(i)$ is the set of dwelling types that are preferred above type n by households of type i. Furthermore, use will be made of a variable δ:

$$\left.\begin{array}{l} \delta_{nm}(i) = 1 \text{ if } m \varepsilon S_n(i) \\ \delta_{nm}(i) = 0 \text{ if } m \varepsilon S_n(i) \end{array}\right\} \quad (3.5)$$

With the aid of this variable the number of households of type i moving from a dwelling of type n into one of type m can be described as:

$$s_{nm}(i) = (1-v)\, q\pi_m . \delta_{nm}(i)\, B_n(i) \quad (3.6)$$

$$n \neq m;\ n, m = 1, \ldots, N+1$$
$$i = 1, \ldots, I$$

The total stream of households moving from type n dwellings into type m dwellings can be obtained from (3.6) by summation over i. Equation (3.6) is the counterpart of (2.10) in section 2.

It may be concluded that it is possible to extend the model to the case in which different types of households exist: the streams of removers can be described in a way analogous to the standard model. It will however not be easy to compute the stationary state values of the $B_n(i)$'s in an analytical way since in the extended model it will in general not be easy to determine the number of searchers. (In the standard model this was equal to $B-B_1$ and B_1 could be computed easily). In the present, extended version of the model, non-searching households may exist in every type of dwelling, giving rise to more complex fomulas.

As an example of the ideas above, one might consider a housing market with 3 housing types (N=3) ranked in decreasing order of attractiveness; and two household classes (I=2), where i=1 and 2 indicated higher and lower income households, respectively. Then a <u>dual</u> housing market with limited interaction between the market segments can be characterized as follows. Assume that dwelling types are ranked as follows for higher income households: $U^1 \geq U^2 \geq U^4 \geq U^3$, implying that starting households (n=4) will not accept dwelling type 3. This results in $\delta_{4,1}(1) = 1$, $\delta_{1,4}(1) = 0$, etc. Further, assume that for lower income households the following ranking prevails: $U^2 \geq U^3 \geq U^4 \geq U^1$, which implies that high quality dwellings are infeaasible for this group. This results in: $\delta_{4,2}(2) = 1$, $\delta_{2,4}(2) = 0$, etc. In this model, low quality dwellings (n=3) are only for low income households, high quality dwellings (n=1) are only for high income households. For medium quality dwellings there is competition between the two household classes. Developing this simple model along the lines of section 2 leads to an interdependent system of non-linear dynamic equations of which the stationary state cannot be found in an analytical way so that a simulation approach is appropriate.

Note that up to now the costs of moving are assumed zero. Of course, this assumption can be removed by assuming that households only move when the difference in utility between an offered and the present dwelling exceeds a certain critical level. This will lead to a reduction of the number of non-zero δ's in (3.5) and thus to an overall reduction in residential mobility.

3.2. Stochastic preferences

In the foregoing subsection it was assumed that households with the same characteristics and the same income do have exactly the same tastes with respect to the dwelling types. One may doubt whether this is a realistic assumption: households of the same composition and with the same income may nevertheless have different tastes, although this variability becomes smaller as more and more characteristics will be taken in consideration. For practical purposes it will not be possible to increase the number of relevant characteristics to infinity and for this reason some variability in tastes may have to be taken into account.

This may be done by assuming that the utility that a households of type i experiences when it occupies a dwelling of type n is the sum of a determinate and a stochastic part:

$$V^n(i) = U^n(i) + \varepsilon^n(i) \tag{3.7}$$

In this equation $V^n(i)$ denotes total utility experienced, $U^n(i)$ denotes the determinate part and ε the stochastic part. The stochastic term $\varepsilon^n(i)$ is assumed to have expectation zero.

The probability that a household of type i will be willing to move from a dwelling of type n to one of type m is equal to:

$$\begin{aligned} p_{nm}(i) &= \text{Prob}[V^m(i) > V^n(i)] \\ &= \text{Prob}[U^m(i) - U^n(i) > \varepsilon^n(i) - \varepsilon^m(i)] \end{aligned} \tag{3.8}$$

If stochastic preferences are introduced in the model the equations for the streams of of removers become:

$$s_{nm}(i) = (1-v)\, q\pi_m p_{nm}(i)\, B_n(i) \tag{3.9}$$

Comparison with (3.6) shows that in (3.9) $p_{nm}(i)$ is used instead of $\delta_{nm}(i)$. Clearly, stochastic preferences lead to two-way traffic even for households of the same type, so that one may expect higher degrees of total residential mobility compared with the case of deterministic preferences.

The introduction of stochastic preferences removes the discontinuities that are associated with the use of $\delta_{nm}(i)$'s. Small changes in a parameter (e.g. the price of the n-th type of dwelling) may cause a jump from δ_{nm} from zero to one or in the reverse direction. In the situation with stochastic preferences, however, small changes in prices may be expected to cause only small changes in the probabilities $p_{nm}(i)$.

Another advantage of the use of a stochastic formulation of preferences is that it relates the present model to discrete-choice-models (e.g. the logit model) that are widely used nowadays to analyse behaviour on the housing market (cf. Clark and Van Lierop, 1986). Note that when stochastic preferences are used in a dynamic context (e.g. by means of simulation models), it is usually assumed that the stochastic part of the utility function is independent between time periods. In that case, the stochastic part can only be thought to represent factors of temporary nature such as incidental changes of income or preferences.

3.3. Social mobility

The introduction of different types of households into the model may be viewed as a major step towards reality. Changes in household characteristics and incomes, together with the existence of different types of households with different preferences are a major determinant of mobility on the housing market. It will therefore be useful to introduce changes in household characteristics and incomes (i.e. transitions of households from one type to another) in the present model.

In order to do this we introduce age-specific transition matrices $T(l)$ the elements of which $t_{ij}(l)$ are the probabilities that a household presently in class i and of age l will in the next period will be of class j (it will then of course also be of age l+1). The age speficity is introduced here explicitly in order to facilitate the modelling of the demographic process that takes place jointly with the social mobiliy. (Note that the interpretation of the index i in section 3.2 differs somewhat from the one presently adopted: in section 3.2 it was wider and could also contain age as a determinant). The following equations may now be formulated (see also Rogers, 1975).

$$B^t(i,l) = (1-v_{l-1}) \sum_{j=1}^{I} t_{ji}(l-1)\, B^{t-1}(j,l-1) \qquad (3.10)$$

The transition probability t_{ii} is defined as $1-\sum_{i\neq j} t_{ij}(l-1)$, the probability of remaining in the same class.

When social mobility and ageing are taken into account in this way, the equations for the streams of households become:

$$s^t_{nm}(i,l) = q^t \pi_m\, p^t_{nm}(i,l) \sum_{j=1}^{I} (1-v_{l-1})\, t_{ji}\, (l-1)\, B^{t-1}_n(j,l-1) \qquad (3.11)$$

$$n \neq m;\ n=1,\dots,N+1;\ m=1,\dots,N;\ i=1,\dots,I;\ l=0,\dots,L$$

New cohorts have age zero and are assumed to occur in fixed amounts: the values $B^t_{N+1}(j,0)$ are assumed to be exogenously given in such a way that population remains constant.

Equation (3.10) introduces the possibility to take life-cycle phenomena into account in the model: starting households will usually consist of one or two persons with a relatively small income and will gradually change both in size and income and therefore in preferences over types of dwellings.

In principle the transition probabilities may be related to explanatory variables, but a discussion of this topic would take us too far away from the subject of this paper.

3.4. Conclusion

The stock-flow model implied by (3.11) is a fully interdependent non-linear model. Ultimately, the probabilities q, p and π depend on the distribution of households over the stock of dwellings. The probability of receiving an offer depends on the vacancies of each type and the number of searchers, but these will in turn be determined by the distribution of households over the different types of dwellings.

The stocks resulting from the flows $s_{mn}^{(t)}$ (i,l) can be formulated as follows:

$$B_n^t (i,l) = \sum_m s_{mn}^t (i,l) - \sum_m s_{nm}^t (i,l) + \sum_{j=1}^{I} (1-v_{l-1})\, t_{ji}\, (l-1)$$
$$B_n^{t-1} (j,l-1) \qquad (3.12)$$

for all relevant values of n,t,i and l.

One may wonder whether anything can be said about the existence and stability of stationary states of the present model. In the case in which π, q and p are continuous functions of the B_m^{t-1} (i,l), the existence of a stationary state is guaranteed by the Brouwer fixed-point theorem (see Appendix A). It is however not easy to say something on (local) stability, since this depends on the values of the characteristic roots of the system (3.12) and these are hard to obtain in an analytical way. Therefore, the proporties of the extended housing market model will be investigated by means of some stimulations in the next section.

4. Simulation experiments

The purpose of the present section is to illustrate the working of the extended model described in section 3 by means of some simulation experiments. These experiments serve two purposes. First, they enable one to investigate whether the system converges towards a stationary state, although of course this does not really prove stationarity in all possible situations. For the experiments described below, in all cases a stationary solution was found. Second, they allow one to study the effects of changes in exogenous variables (and parameters) on the stationary state. Thus, one would be able to predict the effects of changes in policy (or autonomous variables) on the housing market.

4.1. Description of the basic situation

In order to provide meaningfull results we looked for an initial situation that would give a good impression of the potential of the model but that would at the same time be simple enough to allow a clear interpretation of the results.

The total amount of households, as well as its composition over the different age groups and social classes is constant, i.e. a stationary population is used as a starting point.

A simple way of taking account of the life cycle is to introduce small and large households. All starting households are small. As time passes they grow in size and finally they shrink again to small households. The creation of new households is related to the shrinking of the older ones.

It is assumed that households exist for a period of at most 40 years although most of them will disappear before that high age is reached. During the first 10 years households grow in size from small to large. In the second period of 10 years they remain large and in the third decade of their existence they shrink to small size again. At the same time new households are created, i.e. children become adults and create households themselves. During the last 10 years of their possible existence the old households gradually disappear.

Small households prefer small dwellings and large households large ones. In order to complicate the situation a little bit more two social groups are distinguished although for the time being we assume that no social mobility takes place between them. Thus, there are small and large households of two social groups. Dwellings are distinguished by the preferences that the different kinds of households have for them. There is one type of dwelling especially preferred by households of small size of the first social group, one especially preferred by those of big size of the second social group etc.. All households are indifferent to the type of dwelling they rent when it is not their most preferred type. However, a dwelling of whatever kind is preferred to none. Preferences are, for the time being, assumed to be deterministic (see section 3.1.).

In fact the assumptions mentioned above introduce a dichotomy into the housing markets. There are two submarkets, one for each of the groups. The dichotomy is not complete, however, since starting households will accept any offer and thus may rent a type of dwelling that will be higher preferred by households of the other social group. Later on we will introduce social mobility between both groups.

Allocation is assumed to take place four times a year. This implies that we had to work with 160, rather than 40, periods. Searching households will get an offer of a vacant dwelling at most four times a year. The offer, the decision to move and the actual move are assumed to take place within a period of 3 months.

Two types of searching households can be distinguished: the starting ones, who will accept any ofer they get, and those not in a type of dwelling they prefer most, who will accept only a dwelling of their most preferred type.

In every quarter 50 new households are created, while in the same period 50 old ones disappear, so that the total number of households is constant. The polulation structure is summarized in Table 2.

	Age groups of households (years)				
	0-10	10-20	20-30	30-40	total
social group 1					
small	352	0	434	194	980
large	648	959	289	0	1896
social group 2					
small	352	0	434	194	980
large	648	959	289	0	1896
total	2000	1918	1446	388	5752

Table 2. Summary of the stationary population.

As a starting situation we use as values for the total amount of dwellings 900 for both types of small households. Since there are 980 small households and 1896 large ones in each social group there exists a moderate overall excess demand on the housing market. In the stationary state this configuration gives rise to occupancy of 94 % of the small dwellings and 97 % of the big ones. The average duration of residence, measured as the ratio of the total number of households occupying a certain type of dwelling and the total outflow from that type of dwelling (i.e. the total number of removing and disappearing households), is approx. 8.7 years for the small dwellings and almost 16 years for the large ones. This measure provides an overall picture of the duration of stay. Expected durations of stay for the different age groups, household sizes and social classes may differ widely from the average figure.

The average duration of vacancy is measured as the ratio of the total number of vacant dwellings and the total inflow of households and amounts to approx. 0.6 years for small dwellings and 0.5 years for large ones.

Vacancy chains, measured analogously to the way described is section 2, are equal to 2.05 for small dwellings and 2.16 for large ones.

Starting households do on average have to wait almost 2.4 years before they get an offer. The probability of getting an offer is equal to some 10 % per quarter, which amounts to approx. 1/3 per year. This seems to be fairly low, and justifies policy measures for this group.

Most of the households will in the start of their life cycle not obtain a small dwelling of their most preferred type, but after 10 years 50 % of the households, that have all become large ones by then, has a dwelling especially preferred by large households of their social group. This percentage increases to 82 % after 20 years. In the next ten years all large households become small again. In their 30th year 47 % of the households live in a small dwelling. During the last 10 years of their existence this percentage increases to 80 % but only a small fraction (1 %) of the households reach their 40th year. Thus it may be concluded that in the market described by the model there exist a lot of unfulfilled wishes as is clear from the total number of searchers, viz. almost 2850 on a total population of 5752. When as much households as possible were allocated in a dwelling of their most preferred type the total number of searchers would have been 252, none of these searchers renting a dwelling. This is less than 10 % of the actual number. Optimal and actual allocation are thus very different from each other, due to market frictions, a situation not unlike those regularly observed in real world housing markets.

4.2. Changes in the dwelling stock

As one would expect the functioning of the market becomes much more smooth as the stock of dwellings is increased. In our second variant (see Table 3) the number of both types of small dwellings is increased to 1000, that of big dwellings to 2000. From Table 3 it can be inferred that in this case there are more vacancies due to the overall excess supply of 250. The duration of residence is smaller because of the greater ability of people to change to a more preferred dwelling (the probability of obtaining an offer during one quarter is now 0.28). The duration of vacancy has not increased much. Vacancy chains however have increased significantly in length. The total number of searchers and the number of households without a dwelling has decreased considerably.

In the present situation of overall excess supply (i.e. the number of dwellings is greater than that of households), an allocation of all households in a dwelling of their most preferred type is perfectly possible, but it is not obtained due to market frictions. A relatively high number of households keeps searching, indicating a poor performance of the allocation machanism.

In a third variant of our simulation exercise a different situation for the segments for small and large dwellings is introduced by setting the stock of dwellings for each type equal to 1375 so that the same total stock of dwellings was obtained as in the first variant. Thus in the submarkets for small dwellings there exists excess supply while in those for large dwellings there is excess demand. As one would expect the functioning of the market becomes more difficult in these circumstances, but no large differences with the situation in variant 1 appear. The duration of residence in small dwellings increases, while that in large dwellings decreases. The total numbers of

Table 3. Summary of simulation results[1)]

	variant 1 group 1		variant 2 group 1		variant 3 group 1		variant 4 group 1		variant 4 group 2	
	small	large	small	large	small	large	small	large	small	large
total no. of dwellings	900	1850	1000	2000	1375	1375	700	1650	1100	2050
number of occupied dwellings	844	1795	904	1908	1290	1337	660	1608	1028	1981
vacancy rate (%)	6.22	2.97	9.58	4.56	6.20	2.77	5.77	2.56	6.56	3.37
mobility rate (% per year)	6.04	3.36	9.72	5.20	4.44	3.64	5.60	2.92	6.12	3.60
duration of residence (years)	8.73	15.84	5.91	14.34	11.00	15.25	8.83	15.89	8.85	15.98
duration of vacancy (years)	0.58	0.48	0.63	0.68	0.73	0.43	0.54	0.42	0.62	0.55
searching households (%)	62.27	34.97	34.13	16.86	68.22	34.96	60.67	35.06	64.44	36.09
length of vacancy chains	2.05	2.16	2.88	2.86	1.93	2.26	2.11	2.22	1.96	2.06
total no. of searchers	2849		1494		3260		2884			
total no. of vacancies	222		374		247		224			
number of households without dwelling	474		126		499		476			
probability of of receiving an offer (per quarter)	0.095		0.214		0.091		0.095			

1) Dwelling types are indicated by the household groups giving them highest preference. In variants 1, 2 and 3 the figures for both social groups are equal. Therefore only those of the first social group are given.

searchers, vacancies and households without a dwelling have increased, but only moderately.

Another variant (the last we want to deal with in this section) of the model is obtained by introducing differences in the submarkets of the different social groups. This is done by setting the number of small dwellings for the first group equal to 700, that for the second group to 1100. The number of large dwellings for the first group becomes 1650, that for the second group 2050. So the first group is confronted with excess demand, the second with excess supply, while the total number of dwellings is equal to that in variant 1.

As becomes clear from Table 3, in this case again only small differences with the numbers associated with variant 1 are obtained. The percentage of searching households in dwelling category 2 is relatively high, because many households not preferring this dwelling type are forced into it because of the lack of other dwellings. The total number of searchers is almost equal to that in variant 1. Clearly the smaller number of searchers of group 2 compensates almost completely for the higher number of searchers of group 1.

4.3. Social mobility

Until this point we have maintained an assumption of zero mobility between both social groups. This assumption will now be relaxed in the following way. Total births per quarter remain equal to 50 but now 35 new households will be of social group 1 and 15 of group 2. During the first ten years of their lives household have a chance of moving from group 1 to group 2. In every quarter 1% of the households of group 1 move to group 2, while remaining equal in size. During the next twenty years mobility occurs in both directons: in every quarter 1% of the inhabitants of both groups move to the other. After reaching the age of 30 years mobility is restricted to moves from group 2 to group 1, again with an intensity of 1% per quarter.

It may be helpful to think of households in group 1 as the relatively poor ones and those in group 2 as the relatively rich ones. Movements form one group to the other one due to chance, but there is a gradual upward movement in income positions during the early stages of the life cycle and a gradual downward move in the last years. Rich starting households are "born" from rich parent households only, while poor starting households can be born from parents from both social groups. Aggregate figures of the population are given in Table 4.

Table 4 Distribution of households according to age and social group

	age group (years)				
	1-10	10-20	20-30	30-40	total
group 1					
small	450	0	427	207	1084
large	709	917	282	0	1907
group 2					
small	254	0	442	180	876
large	587	1002	295	0	1885
total	2000	1918	1446	388	5752

Table 5 presents some figures of variants 5 and 6 corresponding to 1 and 2 resp. in the numbers of dwellings of each type. Since the numbers of small and large households are not equal for both social groups, in the case of social mobility, figures for all four types of dwellings are presented.

The introduction of social mobility between both social groups gives rise to a more dynamic picture of the housing market as is clear from a significant drop in the average duration of residence. Somewhat surprisingly some other variables (e.g. duration of vacancy, number of vacancies) do not change very much. The total number of searchers in the case of a moderate access supply (variant 6 as compared to variant 2) increased significantly. Mobility rates are higher, the length of vacancy chains has increased as has the share of searching households. Summarizing, it may be said that the friction in the market has grown considerably as a result of the introduction of mobility between social groups.

5. Concluding Remarks

In section 2, it has been shown how the introduction of search elements in a regulated housing market model with various dwelling types leads to a non-linear dynamic system, the non-linearity being related to limitations in housing supply. This system has a unique and stable stationary state, each dwelling type being characterized by a different value of duration of vacancy, duration of residence and vacancy rate. This model differs from most usual residential search and mobility studies in that it takes into account that households moving from one dwelling to another leave a vacant dwelling, thus influencing the prospects of future searchers (see e.g. Rietveld,

Table 5. Results of the introduction of social mobility

	variant 5				variant 6			
	group 1		group 2		group 1		group 2	
	small	large	small	large	small	large	small	large
total no. of dwellings	900	1850	900	1850	1000	2000	1000	200
vacancy rate (%)	6.46	3.27	7.08	3.29	9.82	4.98	12.02	5.22
mobility rate (% per year)	6.08	3.92	6.20	3.92	9.68	6.36	9.84	6.44
duration of residence (years)	8.76	14.52	8.61	14.33	6.10	12.00	5.95	11.87
duration of vacancy (years)	.60	.49	.65	.48	.66	.62	.81	.65
length of vacancy chains	2.15	2.30	2.08	2.30	3.12	3.20	2.93	3.15
share of searching househ. (%)	64.81	42.74	67.14	42.35	43.14	27.91	44.99	27.09
total no. of searchers	3198				2120			
total no. of vacancies	243				422			
total no. of households without dwelling	495				174			
probability of of receiving an offer (per quar.)	0.092				0.223			

1984).

In section 3, several extensions of the model have been discussed, especially the introduction of various household types, giving rise to the possibility to study the role of household life cycle effects in the housing market.

Section 4 is devoted to a discussion of several simulation experiments with the extended household model. It appears athat in a situation of overall excess demand - with the given household preferences - the total number of vacancies is rather insensitive to the distribution of supply over the various market segments. Failure to obtain a good match between supply and demand at the various sub-markets will only manifest itself clearly in the form of vacant dwellings when the situation of overall excess demand is replaced by overall excess supply. Even with a very low rate of excess demand on the housing market, the number of households searching for a better dwelling and the number of househods not occupying a dwelling is relatively large given the stochastic matching mechanism of demand and supply. Thus, friction is an important phenomenon for the type of housing market considered.

Future work with the model will be addressed to calibrating the model for a real world housing market. In addition, the model structure can be extended or refined in various directions, for example by an endogenization of construction activities as well as of the number of households (see Snickars, 1978). Also, more refined ways of modeling household formation and termination have to be considered (see Rima et al., 1985). Further, a more refined formulation of search behaviour is in order (Clark, 1982). In addition, an investigation of alternative matching mechanisms between supply and demand would be interesting. Finally, an effort can be made to develop a model for an unregulated housing market along the lines of the present model. This would entail the introduction of endogenous prices into the model.

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Appendix A Existence of Stationary States

We will make use of Brouwer's fixed-point-theorem (see e.g. Debreu, 1959):

If S is a non-empty, compact, convex subset of R^m and if f is a continuous function from S to S, then f has a fixed point.

In this paper $S=\{\underline{B}: 0 \leq \underline{B} \leq W, \sum_{i=1}^{N+1} B_i = B\}$, where $\underline{B}$ is a (N+1)-dimensional vector with elements $B_1,\dots,B_{N+1}$ and $\underline{W}$ an (N+1)-dimensional vector with elements $W_1,W_2,\dots,W_N$, B. Clearly S is a non-empty, compact and convex vector of R^{N+1}/ The mapping f is given by equation (2.12), after substitution of (2.11). It is continuous as long as $B_1 \leq B$, a condition that is satisfied if $W_1 \leq B$, which will be assumed.

A fixed-point is a vector of $\underline{B}$ that is mapped into itself, and therefore a distribution of B over the different types of dwellings that does not change from one period to another. This is what is called a stationary state in the paper and the existence of such a state in the model of section 2 may thus be taken for granted as long as $W_1 \leq B_1$.

The above theorem guarantees the existence of a stationary state also in the extended version of the model given in (3.13), as long as q^t_{nm} and p^t_{nm} are continuous functions of the $B^t_n(i,l)$'s.

It is easy to generalize this argument to the extended models of sections 3 and 4.

Appendix B Analytical Solution of Stationary State in the Basic Model

Equation (17) can be rewritten as:

$$B_n^2 - \left[\frac{v}{1-v} (B-B_1) + \sum_{m=1}^{n-1}(W_m - B_m) + (B - \sum_{m=1}^{n-1} B_m) + W_n \right] B_n +$$

$$+ W_n(B - \sum_{m=1}^{n-1} B_m) = 0 \qquad \text{(B1)}$$

This may be solved for B_n in the usual way:

$$B_n = 1/2 \{c_{n1} \pm \sqrt{c_{n1}^2 - 4c_{n2}}\} \qquad \text{(B2)}$$

where c_{n1} is the expression in square brackets in (B1) and $c_{n2} = W_n(B - \sum_{m=1}^{n-1} B_m)$. c_{n1} is positive and greater than W_n.

The expression $c_{n1}^2 - 4\,c_{n2}$ is positive since c_{n1}^2 contains only positive terms, among which $2Wn\,(B-\sum_{m=1}^{n-1} B_m) + W_n^2 + (B-\sum_{m=1}^{n-1} B_m)^2$ and the sum of the latter two is at least as great as $2W_n(B-\sum_{m=1}^{n-1}B_m)$ since $[W_n - (B-\sum_{m=1}^{n-1} B_m)]^2$ is positive. So it may be concluded that there will be two real roots for B_n. Only the root associated with the minus-sign in (B2) is of relevance however since B_n has to be smaller than W_n, so that $c_1 + \sqrt{c^2{}_1 - 4c_2}$ has to be smaller than $2W_n$:

$$c_1 + \sqrt{c_1^2 - 4c_2} \leq 2W_n$$

$$\rightarrow \quad \sqrt{c_1^2 - 4c_2} \leq 2W_n - c_1$$

$$\rightarrow \quad c_1^2 - 4c_2 \leq 4W_n^2 - 4W_n\, c_1 + c_1^2$$

$$\rightarrow \quad c_2 = W_n(B - \sum_{m=1}^{n-1}B_m) \geq W_n(c_1 - W_n)$$

The last inequality is however contradicted by the fact that c_1 contains only positive terms among which W_n and $(B-\sum_{m=1}^{n-1} B_m)$. It may be concluded therefore that the positive sign in (B2) is never relevant.