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# **SERIE RESEARCH MEMORANDA**

**MODELLING NON-LINEAR PROCESSES IN TIME AND SPACE**

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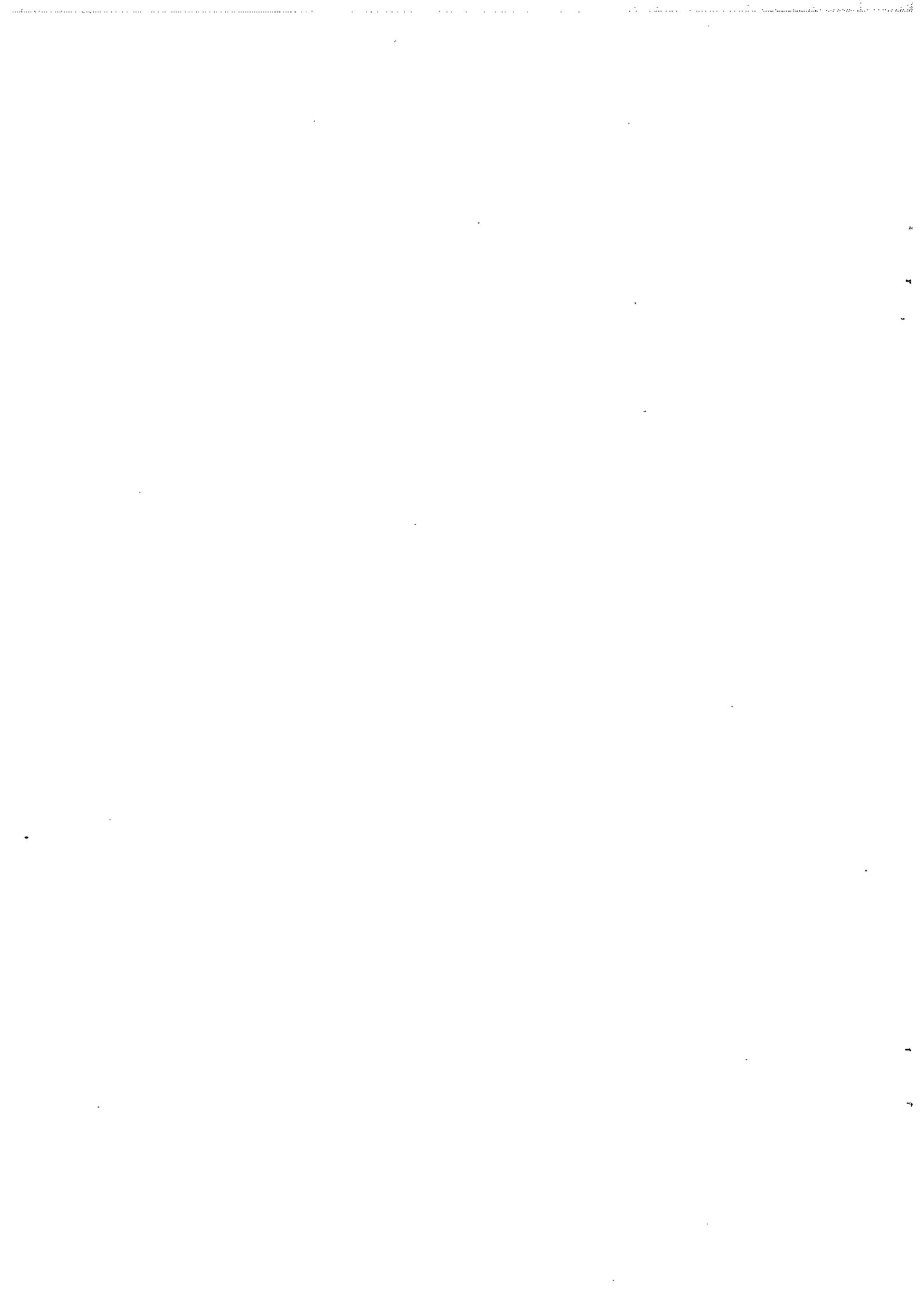


MODELLING NON-LINEAR PROCESSES IN TIME AND SPACE

Wim Barentsen

Peter Nijkamp

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Regional Dynamics, Umea, Sweden, June 1986



## 1. Introduction

In recent years one observes a growing interest among economists in non-linear dynamic systems. Several reasons can be mentioned for this 'upswing' in the attention for non-linear dynamics.

First, after the path-breaking methodological contributions to (in)stability and (dis)equilibrium analysis in the natural science field made among others by Thom (1975) and Nicolis and Prigogine (1977), a stimulus was given to a thorough investigation into the nature of non-linear dynamics in the social sciences (see e.g., Weidlich and Haag, 1983). It was increasingly realized that dynamic interactions between the components of a complex system marked by dissipative structures affecting inter alia the homogeneity of time and space may lead to a large spectrum of evolutionary patterns of such a system (ranging from inert and stable behaviour to fluctuating and unstable behaviour).

In the second place, structural changes in the economic conditions of western societies have caused an increasing interest in non-linear evolutionary patterns. Long-wave patterns in macro-economic and regional systems, long-term drastic shifts in economic activities, and differential dynamic trajectories of various subsystems of the economy have demonstrated the relevance of non-linear approaches in economics. In fact, as soon as parameters of an otherwise linear system are time-dependent with respect to endogenous variables of this system, one faces a situation of endogenously determined structural change leading to non-linear dynamic models. In a space-time context such structural changes may lead to interesting questions regarding spatio-temporal (ir)reversible trajectories (including catastrophic behaviour) of a dynamic system.

Finally, for a long time the mathematical-statistical difficulties inherent in non-linear dynamic systems have precluded many researchers from applying such approaches to social sciences, but the rapid computational advances in this field (including operational computer software) and the availability of an appropriate mathematical framework (notably the analysis of the qualitative behaviour of a dynamic system) have led to an increased use of non-linear dynamic models in economics.

Examples of such applications can be found in various fields of economics, such as macro-economics (e.g., long waves analysis), consumer economics (e.g., shopping behaviour), regional economics (e.g., urban life cycle analysis), and business economics (e.g., technology innovation behaviour). An intriguing question in this respect is the relationship between the micro behaviour of system's components and their macro consequences for the system as a whole: changes at a micro level may - beyond a certain critical threshold level - exert structural influences at an aggregate level. Clearly, linear models are in general unable to generate structural or

sometimes discontinuous changes.

The class of non-linear dynamic models has specific features which distinguishes them from conventional linear dynamic models. The main characteristic is that such models are able to describe qualitative system's changes, as in a non-linear dynamic model there may be certain ranges of parameter values for which the system can be in multiple equilibrium states. Such ambiguity which in any case does exist in a formal sense can only be eliminated if either the theory underlying the non-linear dynamic model specification is made more specific (i.e., more oriented toward the behaviour of the system under these parameter values) and hence more aligned to the phenomena to be studied (to diminish the semantic insufficiency), or if more insight is available on the long-term historical evolution of the phenomenon at hand (requiring full and non-trivial information on past behaviour). In general however, economic theories are semantically insufficient to avoid a priori the possibility of multiple equilibrium states. The existence of various types of feedback mechanisms in economic systems may lead to non-linear trajectories and even discontinuous changes. Such discontinuous changes are often time-irreversible, i.e., by reversing the direction of the initial stimulus that has caused the discontinuous movement (e.g., bifurcation, catastrophe, or shock), the system does not necessarily move back to its original state. Such asymmetric behaviour implies an unstable evolutionary pattern, as the discontinuities which may then be triggered by marginal changes in initial conditions or in parameter values, make the system's evolution time-irreversible. Consequently, the past state of a system plays a dominant and non-trivial role in non-linear dynamic systems.

It is worth noting that non-linear dynamics plays a crucial role in explaining the spatio-temporal evolution of a spatial system (e.g., city, region), as here the question of isotropy of space and time is at stake. The analysis of the development of geographical structures requires an investigation into the existence of reversibility of space-time systems. An abstract representation of a geographical structure can be given by the Cartesian coordinates  $(x, y)$  of the successive phenomena to be modeled in order to position them in a two-dimensional surface. Additional dimensions (e.g.,  $z$ ) may of course be added to account for other attributes of such a phenomenon, for instance, its size or magnitude, its degree of spatial interaction with respect to other phenomena in space and time, etc. Geographical structure in a general sense then refers to the interrelatedness between locational aspects  $(x_i, y_i)$  and other dimensions  $z_i$  of a phenomenon  $i$ . Clearly, such structures are the result of a historical process (e.g., investment and locational decisions), which might inter alia be described by means of event-history analysis (see Hannan and Tuna, 1985) in a discrete sense, or by means of continuous

space-time models (see Beckmann and Puu, 1985) in a continuous sense. In all these cases the structure and evolution of geographical systems may be analyzed by means of non-linear dynamic models exhibiting discontinuities and irreversibilities. To illustrate the relevance of such approaches, one may quote Griffith and Lea (1983) who remark: "Geographical systems, such as school systems, and geographical networks such as grain elevator and gas station networks, experiencing rationalization, growth or contraction and decline, have demonstrated empirically the asymmetry of life-cycle trajectories".

Paragraph 2 contains some remarks on dynamic systems. Different forms of bifurcation are shortly treated in paragraph 3. In paragraph 4 we distinguish three levels on which dynamic processes can be modelled. These three levels are used to classify some models as have been developed by economists and geographers. Finally, paragraph 6 contains some conclusions.

## 2. Dynamic Systems

Following Samuelson (1948, p. 314) and Frisch (1935-36) we may define a dynamical system, as a system whose behavior over time is determined by functional equations in which variables at different points of time are involved in an essential way. This definition is made more precise in its application to economics by requiring that the variables should be economically significant. Otherwise every variable can be written as the derivative of its own integral, which itself may not be a variable of interest, although it would make the system dynamic. Accounting for this qualification we regard systems of difference and differential equations as dynamic systems. In the sequel of this paper we will only pay attention to these two classes of dynamic equations.

There are various ways of classifying dynamic models. For instance, Samuelson has made a subdivision into complete causally-determined systems, historical systems, and stochastic (historical and non-historical) systems (see for details Samuelson, 1948).

The analytical knowledge of systems of differential equations is better developed than that of systems of difference equations. Several results for systems of differential equations hold however in an analogous way for systems of difference equations. Sometimes however, unexpected results may take place, viz. if differential equations are discretely approximated by means of difference equations. The problem here is caused by the fact that empirical economic data are usually only available at discrete time intervals, so that in economic research practice for dynamic systems one is forced to use difference equations. This problem can be clarified as follows. Assume the following dynamic system:

$$\dot{x} = F(x) \tag{2.1}$$

where  $x \in X \subset \mathbb{R}^n$ ,  $\dot{x} \equiv \frac{dx(t)}{dt}$  and  $F: X \rightarrow \mathbb{R}^n$ .  $X$  is called the state space and  $F$  defines a vector field on  $X$ , while (2.1) is a general formulation of a system of differential equations.

In applications where numerical solutions are required, the distinction between differential and difference equations becomes blurred, as the computer solution of (2.1) usually requires a discrete approximation:

$$\frac{x(t+\Delta) - x(t)}{\Delta} = F(x_t) , \tag{2.2}$$

where  $t = 0, \Delta, 2\Delta, \dots$  with  $\Delta$  a small positive number. Let  $n = 1$ ;

; local stability of the corresponding equilibrium  $x^*$  of (2.1) requires:

$$\left. \frac{dF}{dx} \right|_{x=x^*} < 0 , \tag{2.3}$$

while local stability of the corresponding fixed point  $x^*$  of (2.2) requires:

$$\left| 1 + \Delta \frac{dF}{dx} \right|_{x=x^*} < 1 \quad (2.4)$$

The latter illustrates that a discrete numerical solution procedure for a differential equation system evokes problems of stability: if  $\Delta$  is taken too large, the fixed point  $x^*$  of the approximated system (2.2) may exhibit unstable behaviour, although it may be a stable equilibrium point for system (2.1).

A glaring example of such a situation can be found in models of the type developed by May (1974), which are frequently used in population dynamics (see also Pimm, 1982, Li and Yorke, 1975 and Brouwer and Nijkamp, 1985).

The prototype of the May model has the following form:

$$x(t+1) = \psi x(t)[1 - x(t)] \quad (2.5)$$

This simple non-linear dynamic system in difference equation form may exhibit a remarkable spectrum of dynamic behaviour ranging from stability to fluctuating and even chaotic patterns, depending on the parameter values and on the initial conditions. This unusual and unexpected behaviour of a non-linear dynamic model does however, not hold for its continuous counterpart in differential equation form. Consequently, the conclusion may be drawn that the May model derives its unusual results mainly from its specification in difference equation form. Although such models may generate a wide spectrum of dynamic behaviour, there is a priori no reason to believe that simple models of this type are able to provide a more realistic and reliable representation of a dynamic complex world than other models would do.

With regard to models of type (2.1) it is interesting to pay attention to 7 fundamental questions raised inter alia by Varian (1981):

- (i) Do solutions exist?
- (ii) Do equilibria exist?
- (iii) What is the number of equilibria?
- (iv) Which equilibria are locally stable?
- (v) Which equilibria are globally stable?
- (vi) Do cycles exist?
- (vii) Is the system structurally stable?

These questions will concisely be treated here in order to clarify some relevant aspects of (non-linear) dynamic models.

A solution to (2.1) - with initial conditions  $x(0) = x_0$  is a differentiable

function  $x : I \rightarrow X$ , where  $I$  is an interval in  $\mathbb{R}$ , such that:

$$\frac{dx(t)}{dt} = F(x(t)), \text{ and } x(0) = x_0 \quad (2.6)$$

If  $F$  is continuous differentiable on the open subset  $X$ , a unique solution does exist. This solution is continuous (as a function of  $x_0$ ). An explicit analytical solution to (2.1) is however usually difficult to find, so that normally the attention is focussed on the qualitative properties of this system. In this framework, the issue of the existence and the stability of equilibria of the system emerges.

An equilibrium is defined as a point  $x^* \in X$  such that  $F(x^*) = 0$ .

Various theorems dealing with the existence and number of equilibria - especially in case of non-linear dynamic models - may be relevant in concrete situations (see, for instance, Rijk and Vorst, 1982). Particular important in this context is the notion of local stability. An equilibrium point  $x^*$  is called locally (asymptotically) stable, if there is some  $\epsilon > 0$  such that for all  $x_0$  for which  $|x_0 - x^*| < \epsilon$  it follows that  $\lim_{t \rightarrow \infty} \varphi_t(x_0) = x^*$ , where  $\varphi_t(x_0)$  is the flow of the differential equation (2.1) that corresponds to the initial condition  $x(0) = x_0$ . Usually, only locally stable equilibrium points are regarded as relevant from an economic point of view. Stable equilibria act as attractors of the trajectory of a dynamic system and thus determine a specific solution. The case of unstable equilibria is also interesting, as such points may act as repellers of the trajectories. Finally, a third type of equilibrium is the saddle point which reflects a special kind of instability: a saddle point implies that there are two trajectories leading to different equilibrium points; such a point divides the state space into two areas, while each trajectory in each area is directed towards a different stable equilibrium. The nature of an equilibrium point  $x^*$  can in principle be evaluated by analyzing the eigenvalues of the Hessian matrix  $DF(x^*)$  (see Annex 1).

An equilibrium point  $x^*$  is globally stable, if  $\lim_{t \rightarrow \infty} x(t) = x^*$  for any initial condition  $x_0$ . Further contributions to the analysis of global stability in economics can be found in Arrow and Hahn (1971).

A special type of equilibrium is a cyclical pattern: a point set  $X$  is in a cycle (closed orbit), if  $F(x) \neq 0$  and  $\varphi_t(x) = x$  for some  $t \neq 0$ . Casti (1985) indicates why cycles are relevant in modelling real-world phenomena. In his view, empirical evidence indicates that for many phenomena periodicity is the rule and static equilibrium the exception. Besides, he states that a system that can respond more swiftly to the environment than its neighbours has a competitive advantage. That real-world systems do not always exhibit truly periodic behaviour is due to perturbations which continually push the system from one cycle toward another.

It is worth noting at the end of this section that the equations of a model have

always an approximate character, so that structural stability is a desirable property, i.e., a small change (perturbation) in  $F(\cdot)$  should not change the qualitative nature of the vector field. Structural stability is thus directly related to the 'behaviour' (in terms of location, existence and character) of the equilibrium points. Clearly, under specific circumstances with dramatic or discontinuous changes in a real-world system one may use modelling experiments based on structural instability. In the latter case, linear models with endogenous changes cannot be employed.

### 3. Non-linearity and Bifurcations

Linear model specifications have become very popular in economics, although there are a priori no strong theoretical arguments in favour of linear models. Clearly, practical reasons may be relevant in this context (for instance, data availability, econometric estimation and test procedures, first-order Taylor approximations, computer software like linear programming etc.), but as noted before there are many convincing examples that demonstrate the inadequacy of the use of linear models for various real-world phenomena.

Non-linear models are able to generate non-trivial changes in a dynamic system (not just growth or decline), and consequently the (event) history of phenomena plays a crucial role in dynamic modelling efforts for such a system. The capability of non-linear models to describe and/or to endogenously generate bifurcations is their major discriminating feature. For instance, by means of bifurcation analysis one may try to model structural change processes: the qualitative nature of the change caused by a bifurcation reveals the evolution of the system concerned.

Consider system (2.1) in a slightly modified form:

$$\dot{x} = F(x;\alpha) , \tag{3.1}$$

where  $\alpha$  is a parameter vector. Let  $F$  be non-linear. Assume now that  $\bar{x}_i$  ( $i=1, \dots, I$ ) are  $I$  distinct equilibrium points. The number  $I$  may then be co-determined by the numerical value of  $\alpha$ . Since  $\dot{x} = 0$  for any point  $\bar{x}_i$ , there exists an internal consistency within the system for these values of  $x$ . A state is internally consistent, if it is self-sustaining. The non-linearity of  $F$  may thus lead to a number of distinct self-sustaining states. Existence of cycles reveals that there may also be an internal consistency between moving variables. If one takes for granted that a social system can be decomposed in individuals whose behaviour influences the system's environment and is in turn influenced by this environment, it is easily seen that multiple self-sustaining states may exist.

Each individual action or each local intervention in a complex system may lead to an aggregate impact at the system's level that - after a bifurcation - may result in global changes. Various forms of bifurcation may be distinguished, some of them will be discussed here. Assume for instance the following partition of (2.1)

$$\begin{aligned} \text{(a) } \dot{x}_1 &= f_1(x_1, x_2; \beta) \\ \text{(b) } \dot{x}_2 &= f_2(x_2; \gamma) \end{aligned} \tag{3.2}$$

where  $x_1$  and  $x_2$  are vectors of fast respectively slow moving variables, and  $\beta$  and  $\gamma$  parameter vectors. The functions  $f_1$  and  $f_2$  are assumed to be non-linear.

Since  $f_1$  is non-linear, there may be some ranges of  $x_2$  and  $\beta$  for which (3.2.a)

has multiple equilibria, and there may be some bifurcation points  $(x_2^b, \beta^b)$  at which this number of equilibria may even change. If the system is situated near such a point, it depends on infinitesimal changes in  $x_2$  and/or  $\beta$  which equilibrium point will be reached (note that  $x_1$  is assumed to be a fast moving variable). Consequently at such points  $(x_2^b, \beta^b)$  the self-sustaining nature of the equilibrium points becomes unstable: a small change in  $x_2$  and/or  $\beta$  may trigger a fast development whereby  $x_1$  takes on an entirely different equilibrium value. In such cases, the predictability with regard to  $x_1$  will be low, even if the functional form  $f_1$  were exactly known.

Another example of bifurcation concerns the nature of the equilibrium points, which may alter in response to a small parameter change.

Both types of bifurcation (i.e., those related to the number and nature of equilibria reflect essentially situations of structural instability; they may even occur simultaneously.

A final and different form of bifurcation is related to small changes in the initial conditions. A small shift in the initial conditions may force the system to move to a completely different trajectory. This divergence is caused by the shift in the influence of either attracting points (in case of stable equilibria) or repelling points (in case of unstable equilibria). The treatment of unstable system behaviour requires often a probabilistic approach as will be indicated hereafter.

#### 4. Three Levels of Modelling Dynamics

Models can be designed and estimated at different levels of aggregation, ranging from macro to micro levels. Each aggregation level imposes certain constraints on model specification and validation, as well as on the conclusions to be inferred from model results (see also Blommestein and Nijkamp, 1985). For instance, when a model is specified and estimated as a macro model, one has to be extremely cautious in drawing conclusions on the micro behaviour of economic agents (the so-called problem of 'ecological fallacy').

Each scale of aggregation implies that certain aspects are left out of consideration or are assumed to be constant. For instance, for certain modelling purposes (e.g., short-term forecasts) one may abstract from the explanation of slow moving variables (by using a *ceteris paribus* clause). On the other hand, in other cases one may be willing to neglect the impact of fast moving processes, for instance, if these processes are assumed to be so fast that the analysis is not distorted of the trajectory of these variables from one equilibrium to another is not precisely studied (e.g., in a comparative static framework). Clearly, the validity of these simplifying assumptions depends on the differences in time scales and order of magnitude of the variables involved. However, in a dynamic model fast and slow dynamics may occur simultaneously.

It is worth noting however, that the rate of change of a variable in a model depends usually also on the variables which are left out of consideration. The behaviour of such omitted variables depends in turn on other variables (either included in the model or omitted) etc. Consequently, from the viewpoint of specification analysis, it has to be observed that a closed set of equations (i.e., a finite set of equations specified only in terms of agreed-upon variables) does only exist in an approximative sense.

There are however, phenomena for which the distinction according to time scale and size does not provide a useful way of demarcating a closed set of equations. For instance, in case of structural change processes, we can imagine that a small change at the micro level of a system (e.g., the construction of a road or the introduction of a new production process) may have substantial impacts on the macro level. In these cases "... the difficulty comes from the fact that couplings may exist at all scales from the smallest initial ones to those of the macroscopic level when a system is about to topple over from one stable mode of operation to another" and thus "... a model or theory for describing systems near these critical points must therefore take all these correlations into account in one way or another" (see Courtois, 1985, p. 593).

If the abovementioned new mode of operation is qualitatively different from the

original one, one can speak of an evolutionary event (see also Johansson and Nijkamp, 1986): a small change at the micro level may sometimes change the structure of the system. For instance, in case of a new road the transportation cost matrix for a whole spatial system may change, or in case of a new production process new firms competing with better products, may enter the market. Such new macroscopic structures emerging from microscopic events would of course in turn have an impact on the structure and the functioning of microscopic mechanisms.

After these remarks, we will now discuss three levels of description of change in physical systems as distinguished by Prigogine (1981):

- (i) a macro-phenomenological level
- (ii) a micro-stochastic level (usually based on Markov processes)
- (iii) an approach based on the dynamic laws corresponding to a basic (micro or meso) level.

The fact that in both physical and social systems we can distinguish a micro level and a macro level (structure) may provide a useful analogy. We will therefore use this distinction to classify types of non-linear dynamic models that have been applied in (regional) economics. Thus the aim is not to propose unambiguous design and specification principles for such models; such questions are co-determined by the nature of the phenomenon under consideration, the specific research questions posed etc.

(i). macro-phenomenological level

The variables in the macro-phenomenological approach are (weighted) average values of micro variables whose fluctuations are supposed to have little impact on the first mentioned variables. Clearly, this assumption is not always warranted, witness the occurrence of bifurcations in real world systems deteriorating the macroscopic description. In case of a bifurcation (which ultimately always stems from the micro level), it is clear that complementary theoretical considerations, not included in the macroscopic viewpoint, are needed in order to adequately analyze the bifurcation process and the way the system reorganizes itself.

(ii). micro-stochastic level

In the micro-stochastic approach the micro variables underlying the macro behaviour are explicitly modeled. The factual knowledge regarding their behaviour is limited, so that often a probabilistic approach is followed (e.g., by describing the state transitions of micro variables by Markov processes). By assuming an initial probability distribution for the state of the system, it is then possible to trace the consequences of the micro behaviour and to derive a stationary distribution function.

This distribution reflects then the fact that the multitude of individual events taking play simultaneously at the micro level of the system compensate each other statistically, so that they may create a certain macro order which is called a structure (cf. also the notion of entropy in a spatial interaction system; see Wilson, 1970).

Under certain conditions the stationary distribution may be multimodal, viz. if - given an initial unimodal distribution and its ensuing trajectory - certain points emerge over time at which the distribution shifts from a unimodal to a multimodal one. Such a transition is accompanied by large fluctuations in the micro variables. It is in principle possible to derive approximate mathematical expressions for the growth in such fluctuations proceeding the occurrence of a bifurcation. These fluctuations reflect the existence of a certain ambiguity in the system, as the system may 'choose' between various regimes. Beyond such a bifurcation point the average value of the variables is no longer directly related to the extreme points. Then a multimodal stationary probability distribution results, which indicates that there may be various macro structures that are consistent with the stochastic behaviour of the micro variables. If this micro behaviour is represented by means of a parametrized model, the form of the stationary distribution function may drastically change due to a (small) shift in one of the parameters. As a description of a relatively large system in terms of a probability distribution is often not very meaningful, one moves usually to mean value equations. It should be noted however, that in case of a multimodal stationary distribution the relationships between extreme points and mean values become blurred. Fortunately, it can be demonstrated that the stable equilibrium points of the mean value equations correspond to the extreme points of the stationary distribution. When the behaviour of the micro variables depends on the macro state of the system, the mean value equations will be non-linear and there may be multiple stable modes of operation.

(iii). dynamic laws at a basic level

The third approach provides a description in terms of the dynamic laws operating at a basic level (e.g., the individual trajectories of molecules in a physical system, or the dynamic behaviour of individuals or firms in an economic system). Clearly, the precise demarcation of a basic level implies some arbitrariness. Instead of providing a sharp demarcation criterion for cases (i) and (iii), it is more meaningful to spell out the consequences of not using a phenomenological approach, i.e., in what sense does the analysis change if, in one way or another, one takes into account the fluctuation, diversities and feedbacks at the micro level?

Let us assume that the exact dynamic laws governing the behaviour of basic variables are known. Such laws may express stable behaviour, so that neighbouring

points are transformed into neighbouring points. Then we may use these laws to analyze the behaviour of the system. However, if these laws reflect unstable behaviour, a problem arises, as then any region in the state space, whatever its size, always contains different trajectories that diverge as time passes by. In such cases, even small differences in initial conditions may be amplified. Since a specific limit transition in which processes of a region in the state space are restricted to a point (and hence to a well defined trajectory) is not possible, the description in terms of trajectories breaks down. Then a description in terms of bundles of trajectories becomes relevant. This can be modelled by representing the dynamic equation in stochastic form. This approach, called in the natural sciences the ensemble standpoint, is based on a probability aggregate, which is composed of an ensemble of copies of the original system that are consistent with the information assumed about the original system.

The above mentioned classification of dynamic models is not only relevant for the natural sciences, but also for the social sciences. Especially in disaggregate models of socio-economic behaviour various kinds of non-linearities are likely to exist. Such non-linearities can make the dynamic behaviour of the pertaining model unstable.

In the next section, it will be demonstrated that various models which in the past years have been developed to describe non-linear evolutions, especially of spatial (urban or regional) socio-economic systems, can be classified by means of the foregoing three categories.

## 5. A Classification of Major Types of Non-linear Dynamic Spatial Models

In the past years, various models have been developed by economists and geographers to describe non-linear dynamic (sometimes irreversible) socio-economic developments in time and space. Instead of providing an exhaustive survey of the literature, we will present for each class of non-linear dynamic models discussed in section 4 one or two prototype models which may be regarded as representative for a broad class of models.

### (i) macro-phenomenological models

Various models in this class are directly or indirectly based on the so-called Volterra-Lotka approach in population dynamics. An example of the macro-phenomenological approach to urban growth and form can be found in Dendrinos (1984) and Dendrinos and Mullally (1985). The central point in this approach is that - despite the complexity of a system at the micro-level - it is possible to gain basic insight into the nature of urban evolution by means of an analysis of a limited number of strategic macro-level variables. Dendrinos refers to May (1971) who has shown that - in case of random connectance - a system is more likely to be stable when it is small, the elements of the system are weakly connected and the average strength of interaction is low. Although he admits that inter-city linkages are highly non-random, he uses nevertheless this argument (together with the analytical intractability of large non-linear dynamic models) to reject the use of large-scale models for describing urban evolution in the U.S. which over a time-span of a century has shown a remarkable stability.

He introduces then the concept of an effective environment, which allows him to design a small model for analyzing the income-population dynamics relative to that of the environment of the SMSA's in the U.S. An effective environment is a system's environment which implies such a normalization of variables that their dynamics can be described by means of the Volterra-Lotka dynamics (or any other non-random dynamic model) which provides theoretical insight and makes empirical verification possible (see Dendrinos, 1985, p. 68).

The standard form of this model is as follows:

$$\left. \begin{aligned} \dot{x}_t &= \alpha(y_t - 1)x_t - \beta x_t^2 \\ \dot{y}_t &= \gamma(\bar{x} - x_t)y_t \end{aligned} \right\} \quad (5.1)$$

where  $x_t$  is the relative population size of a metropolitan area (normalized with respect to the total national population size) at time period  $t$ ,  $y_t$  the ratio of urban real per capita income to the prevailing national average during each time period  $t$ , and  $\bar{x}$  the carrying capacity (in terms of population size) of the metropolitan area concerned.

This model structure is claimed to be supported by empirical evidence. The model has to be regarded as a specimen of the macro-phenomenological type, as relative income and population dynamics of an urban area could in principle equally well be modelled in terms of disaggregated variables (such as employment in specific industries, the presence of housing, infrastructure etc.).

The insight which the model may provide is rather limited. The behaviour of an urban system can not realistically be explained without regard of its environment. Despite the normalization used, no functional relationships of the city and its environment are considered. One of the more interesting topics in regional and urban economics is the question: which are the determinants of the relative carrying capacity of an urban area and which are the driving forces of urban dynamics? The analysis however provides no answer to these questions. The relative carrying capacity level of population is not a variable determined endogenously in this approach but it is treated as a parameter for which it is claimed that robust estimates are obtainable. The robustness of these estimates is explained by means of the metropolitan areas having sticky ties to their environment. A relevant point to be explained by a dynamic theory is why this is the case.

One may interpret the abovementioned model as being obtained by implicitly simplifying a more comprehensive model. The validity of explaining a complex phenomenon by means of a few strategically placed macro observations is directly related to the existence of an effective environment, which itself is a somewhat vague concept. It is questionable whether there are many phenomena for which this is possible. In fact, low dimensional models may be useful within a more comprehensive, disaggregated analysis. In the words of Casti (1985, p. 213): "it has been empirically observed in many modelling exercises that the essential behavioral properties of a system which involves interactions of many variables can be captured by centering attention upon a small number of macro-level variables formed, generally, as some (usually non-linear) combination of micro-variables. Usually, the observed macro-variables exhibit the characteristic oscillations, bifurcations, etc., and what is needed is some sort of meso-level theory enabling us to translate back-and-forth between the microvariables, which we cannot see or know, and the macro-patterns".

(ii). micro-stochastic models

An example of this type of model is the work of Haag and Weidlich (1984) on inter-regional migration. They employ synergetic concepts to analyse the dynamics and possible stable modes of operation of a system that describes the distribution of a given population ( $N$ ) over a number of regions ( $L$ ). Within their approach the behaviour of the micro variables of the system (i.e., individuals who might or might not migrate to another region) is explicitly considered. It is argued that the hetero-

generality of their behaviour impede a fully deterministic model. Instead, they make a plea for a probabilistic treatment in terms of transition probabilities for well defined states. These transition probabilities provide a Markov chain. They may be modelled by means of a parametrized model. The parameters in such a model are called trend parameters. The resulting stochastic theory describes the system in terms of a probability distribution defined over the possible states of the system.

By means of the transition probabilities it is possible to link (static) theoretical considerations to dynamics, and by means of the so-called master equation, which describes the evolution of the probability distribution over time, a link is provided between the micro- and macro-level. Then an elegant framework for analysing the dynamics of complex systems results. Once the initial conditions and the specification of the transition probabilities are given, the behaviour of the system is completely determined by the numerical values of the trend parameters. These trend parameters are then estimated on the basis of empirical data. In a more extensive analysis they may be explained by socio-economic factors and in this way make the model more appropriate for prediction purposes (see for more details Annex 2).

In the case that the transition probabilities are functions of the macro-state of the system there exists a feedback from the macro- to the micro-level. This feedback may lead to a multimodal stationary distribution function the shape of which may change drastically under certain critical changes of the trend parameters (bifurcation). The states of the system corresponding to the maxima of the stationary distribution are to be interpreted as stationary end-states in which the spatial interaction system has attained a stable mode of operation. In the case of a multimodal distribution there are several stationary end-states. It depends on the initial conditions which one is attained. Under the condition of 'detailed balance', that is local balance of all probability fluxes, it is possible to derive explicitly the stationary distribution function. An often analytically more tractable but less informative representation is by means of mean value equations. These deterministic equations describe how the mean values of the number of people living in the different regions change as the probability distribution evolves over time. In formula:

$$\frac{d\bar{n}_i}{dt} = \sum_{n_i} n_i \frac{dp(n;t)}{dt} \quad (5.2)$$

where:  $n' = (n_1, \dots, n_i, \dots, n_j, \dots, n_T)$ , a vector consisting of the number of people living in the various regions.

$\bar{n}_i$  = mean value of the number of people living in region i.

$\frac{dp(n;t)}{dt}$  = time derivative of the distribution function concerned.

Bij an approximation which is valid as long as the distribution remains narrow and

unimodal, a closed (self-contained) set of  $L$  differential equations is obtained<sup>1)</sup>. These equations are non-linear in case the individual transition probabilities are functions of  $n$ . The stationary points of the mean value equations correspond to the states at which the stationary distribution attains its maxima. Possible bifurcation phenomena are also reflected in these mean value equations. It depends on the initial conditions which of the stationary points is finally attained by the mean value equation.

(iii). models based on dynamic laws at a basic level

An example of the disaggregated approach belonging to the third class is the work of Allen and Sanglier (1981). Their work is part of the investigation of 'self-organizing' phenomena in natural and social systems. In this approach the interactions of geographically distributed sites can be analysed. It leads to dynamic equations in which various types of non-linearities are present, and consequently the dynamics expressed in these models may reflect an unstable behaviour.

A typical example of these equations reads:

$$\frac{dx_i}{dt} = bx_i(J_i^0 + \sum_k J_i^k - x_i) - mx_i + r[\sum_{j \neq i} \{x_j^2 \exp(-\beta d_{ij}) - x_i^2 \exp(-\beta d_{ij})\}] \quad (5.3)$$

where:  $x_i$  - population of site  $i$

$J_i^0$  - basic 'carrying capacity' of site  $i$

$J_i^k$  - number of jobs in activity  $k$  at site  $i$

$d_{ij}$  - distance between site  $i$  and site  $j$

The parameters  $b$  and  $m$  reflect the demographic change (birth and dead rate) as well as the imobility of the population in relocating residences under pressure from the distribution of available employment. The last term, consisting of a weighted sum of the squared number of people living in the different site, expresses the influence of congestion effects.

The specification of (5.3) which consists of an equilibrium condition and a dynamic adjustment process, is hardly motivated. The dynamics employed is inter alia used in biology (Volterra-Lotka). There it is, under certain circumstances, reasonable to assume that  $x_1 \cdot x_2$  is a suitable proxy (model) for the number of prey-predator interactions between the specimen 1 and 2. It is questionable whether these type of dynamics are appropriate for modelling socio-economic processes. In any case, they imply a type of behaviour that is not motivated.

1) The possible development in case of non-linearity into a multimodal distribution is an extremely slow process which takes place after the distribution has been centred around one of the (ultimate) maxima.

Allen et al. consider spatial structures as being far from equilibrium in a thermo-dynamic sense and consequently they presuppose that flows of matter, persons and energy lead to a maintenance of this disequilibrium situation. The fact that a social system is open is incorporated in the model by two types of stochastic processes that may influence the simulation results in a non-trivial way. One is the random introduction of economic activity at various time intervals at all locations; if certain exogenous boundary conditions are met, the activity will develop, otherwise it will not survive. The second type is concerned with the perturbation of exogenous parameters and the deviations of behavior of an individual agent from an average aggregate performance level. The authors are not so much interested in making exact predictions, as well as in illustrating the consequences of fluctuations and non-linearities. Their explanation is not as 'strong' as the usual 'causal' explanations of classical physics. "It is through the action of elements not explicitly contained in the equations (fluctuations or historical 'accidents') that the choices are in fact made at the various bifurcation points that occur during the evolution of any particular system" and "the spatial organization of a system does not result uniquely and necessarily from the 'economic and social laws' enshrined in the equations, but also represents a 'memory' of particular specific deviations from these average behaviours" (Allen and Sanglier, p. 168).

Another example of the approach based on 'basic' dynamic laws is the type of models employed by Wilson et al. (see for example Wilson, 1981). They have used their models to analyse among other things the urban retail structure and the residential structure. Here we will take the urban retail structure model as an example. In this model two types of agents are distinguished: there are consumers who respond via their buying behaviour to a given spatial distribution of shopping centres and there are entrepreneurs who in response to revenue generated in the different shopping centres determine their investments and consequently the urban retail structure. The buying behaviour of the consumers is modelled by means of a measure of the attractiveness of the different centres and the costs of travel. The attractiveness of the centre is assumed to be a function of the size of the centre. This size itself is determined by the investment behaviour of the entrepreneurs.

The total revenue (expenditure on consumption goods) attracted to centre  $j$  is:

$$D_j = \sum_i \frac{e_i p_i W_j^\alpha e^{-\beta c_{ij}}}{\sum_j W_j^\alpha e^{-\beta c_{ij}}} \quad (5.4)$$

where:  $e_i$  = the average per capita expenditure on shopping goods by the residents of zone  $i$   
 $p_i$  = the population of zone  $i$   
 $W_j$  = the size of centre  $j$   
 $c_{ij}$  = the cost of travel from  $i$  to  $j$   
 $\alpha, \beta$  = parameters

Further it is postulated that the cost of employing a centre is proportional to its size:

$$c_j = kW_j \quad (5.5)$$

where  $k$  is a suitable constant. To analyse the development of the retail structure it is assumed that the entrepreneurs will expand their facility as there are positive profits and that the facility will be reduced in case they make a loss.

So the following equilibrium condition is postulated:

$$\sum_i \frac{e_i p_i W_j^\alpha e^{-\beta c_{ij}}}{\sum_j W_j^\alpha e^{-\beta c_{ij}}} = kW_j \quad (5.6)$$

$$\dot{W}_j = F \left[ \sum_i \frac{e_i p_i W_j^\alpha e^{-\beta c_{ij}}}{\sum_j W_j^\alpha e^{-\beta c_{ij}}} - kW_j \right] \quad (5.7)$$

where  $F$  is a function such that  $F[0] = 0$  and  $F'[x] > 0, \forall x$ . It is worth noting that the specification of the  $F$ -function - in principle to be based on theoretical consideration - influences the nature of the equilibrium points and hence the dynamic trajectory. The model is highly non-linear in the  $W_j$  variables. (5.6) models the distribution of the expenditures over the different centres. The denominator in (5.6) serves to make this distribution consistent. It leads to non-linearities and the possibility of multiple equilibria. The dynamic behaviour of the model may exhibit bifurcation properties that are not easily investigated analytically. This will be even more likely when the model is disaggregated and the attractiveness factors itself become non-linear functions.

After having briefly considered the various prototypes of models belonging to the three abovementioned classes, we will in the final section draw some more specific conclusions regarding the modelling of non-linear dynamic (spatial) systems.

## 6. Conclusion

Non-linear dynamic systems are the appropriate tools for modelling discontinuous and structural changes in real-world systems.

A non-linear dynamic system may, for ranges of numerical values of its parameters, have multiple equilibrium points. So, an equilibrium finally attained may be path-dependent. Non-linearity therefore requires a dynamic analysis and often also a disequilibrium approach. This poses problems. As Koopmans (1957) observed 'until we succeed in specifying fruitful assumptions for the behaviour in an uncertain and changing economic environment, we shall continue to be groping for the proper tools of reasoning'. We are still lacking such a fruitful theory of behaviour in disequilibrium.

The nonlinearity may have its origin in non-linear individual behavioural relationships or may be caused by aggregation of linear ones. However, theoretical knowledge and empirical information concerning functional forms in economics is very limited and the class of non-linear functions is very wide. Conscientious econometric research may be helpful but still requires basic assumptions that may be hard to motivate and will require an enormous effort to test empirically. Comparing different empirical models of a certain phenomenon that are based on different functional forms is far from easy and is not guaranteed to lead to conclusive results. It seems therefore an exacting endeavour to describe specific non-linear phenomena by means of a complete set of well-motivated non-linear dynamic laws.

There arises a dilemma. On the one hand there are empirical phenomena that can not be explained by linear models while on the other hand the knowledge with regard to behaviour in disequilibrium and functional forms is often too limited to warrant the specification of a set of non-linear dynamic laws.

A complete other and in our view promising way of arriving at a non-linear system is the synergetic approach. Synergetics is defined as the science of collective static or dynamic phenomena in closed or open multi-component systems with "cooperative" interactions occurring between the units of the system (Weidlich and Haag, 1983, p. 1). The non-linearity follows from the assumption that the behaviour of the micro-units (the components) of the system is dependent on its macro-state. The macro-state of the system is at the same time influenced by the individual behaviour. There exists a cyclic coupling between causes and effects which may lead to multiple self-sustaining states (structures). The non-linearity enters in a 'natural' way and not by means of an often arbitrary functional form. Since in the synergetic approach the micro- and macro-level are modelled simultaneously it is suited for the analysis of structural change.

The behaviour of the micro-units is modelled by means of transition probabilities

that define a Markov process. Therefore a specific functional form has to be chosen. Modelling the transition probabilities, however, possess a rather well ordered problem, relatively easily accessible to empirical investigation, as opposed to the all embracing nature of the specification of dynamic laws. Within the synergetic approach it is not required to postulate equilibrium conditions and disequilibrium behaviour. Once the transition probabilities are being modelled, the dynamics and equilibria follow in a logical way.

The synergetic approach leads to a description in terms of a distribution function over the state space. This description can be shown to be consistent with a description in terms of stochastic dynamic equations for the relevant variables (see Weidlich and Haag, 1983). In the case that the deterministic part of these equations exhibits unstable behaviour the latter description is rather limited. Simulating the dynamic trajectories for many times will lead to a wide spectrum of results. These results reflect the abovementioned distribution function.

The description in terms of the distribution function is to be preferred since it is more complete. The intractability of the distribution function of somewhat larger system is, however, a serious handicap. Resort is then taken to the mean value equations. These provide in an elegant and concise way (often adequate) information about the distribution function.

In retrospect: the foregoing observations lead us to the conclusion that the power of the master equation is its ability to account for synergetic effects in social systems in an appropriate manner. Further empirical research, amongst others on the actual properties of the mean value equations for real-world systems, is no doubt warranted.

### Annex 1. Stability Analysis by means of Eigenvalues

In this Annex, it will be outlined how equilibrium points  $x^*$  can be dealt with on the basis of an eigenvalue approach of the matrix  $DF(x^*)$ .

The linearized vector field that is close enough to the equilibrium point  $x^*$  describes its stability properties. If at least one of the eigenvalues has a non-zero imaginary part, the convergence or divergence will exhibit an oscillatory pattern. Clearly, when  $F(\cdot)$  is a linear function, there is at most one equilibrium point. In the latter case,  $DF(\cdot)$  is constant, so that then the local behaviour and the global behaviour of the system coincide, besides, the qualitative characteristics of a solution will be independent of the initial conditions. As the linear case implies that  $DF(x^*)$  will generally be a function of the parameters, the equilibrium will change due to a shift in parameters. However, the location of the equilibrium point will not change dramatically in response to a small parameter shift.

On the other hand, if  $F(\cdot)$  is a non-linear function, several equilibrium points may occur; their number depends inter alia on the numerical values of the parameters.  $DF(\cdot)$  is then in general a function of the parameters and of  $x$ . Such a case of multiple equilibria of non-linear models implies that the location, existence and character of equilibrium points may sometimes drastically change in response to a small change in one of the parameters.

Various examples of non-linear dynamic models can be found in regional economics (see also paragraph 5). Several of these models incorporate a static equilibrium condition in a dynamic framework. This equilibrium condition determines the location of the equilibrium points, while their character in addition is also dependent on the dynamic framework. This approach bears a similarity to Samuelson's correspondence principle stating that a fruitful application of comparative static methods often presupposes a theory of dynamics (see Samuelson, 1948).

In conclusion, instead of specifying a priori and in an uncritical way a specific type of dynamics (e.g., Volterra-Lotka dynamics, May dynamics) for economic systems, it is preferable to give sufficient attention to the design of models for dynamic adjustment processes of the dynamic system at hand.

Annex 2. The Master Equation Approach

In this Annex the synergetic concepts as employed by Haag and Weidlich (1984) will be described. The behaviour of the (migrating) individuals is modelled by means of individual transition probabilities  $P_{ji}$  which describe the probability that an individual migrates from region  $i$  to region  $j$  ( $i \neq j$ ). With respect to these probabilities the Markov assumption is made. Thus the probability that an individual migrates from  $i$  to  $j$  in a given time interval is independent of its behaviour preceding that time interval. The probabilities, which link static concepts to dynamics, are modelled as follows:

$$P_{ji}(n_j, n_i) = v \exp[f_j(n_{j+1}) - f_i(n_i)] \quad i \neq j \quad (I)$$

- where:  $P_{ji}$  = individual transition probability for a transition from region  $i$  to region  $j$   
 $v$  = global mobility parameter determining the time scale of the migration process  
 $n_k$  = number of people living in region  $k$   
 $f_k(n_k)$  = utility of an individual living in region  $k$  which has a population  $n_k$ ; the trend parameters that enter this function may be explained by socio-economic factors within the context of a regression model.

These individual transition probabilities enter into the expression for transition 'probabilities' between, so-called, socio-configurations. These socio-configurations are the possible distributions of the given  $N$  individuals over the given  $L$  regions. A typical socio-configuration can be described by the vector  $n$  containing a given number of individuals living in each region. The following model for the transition  $n = (n_1, \dots, n_j, \dots, n_i, \dots, n_L) \rightarrow n^{ji} = (n_1, \dots, n_{j+1}, \dots, (n_i-1), \dots, n_L)$  results:

$$W_{ji}[n] = n_i P_{ji}(n_j, n_i) \quad (II)$$

All the  $n_i$  members contribute the probability  $P_{ji}$ . (II) enters into the master equation, which describes the evolution of the probability that the system is in one of the  $\binom{N}{L}$  socio-configurations.

$$\frac{dP(n;t)}{dt} = \sum_{i,j=1}^L \{W_{ji}[n^{ij}]P(n^{ij};t) - W_{ji}[n]P(n,t)\} \quad (III)$$

The first and second term within the summation express a probability flux that goes into, respectively out of, the socio-configuration  $n$ . Summation gives the net result. Since there are  $\binom{N}{L}$  socio-configurations the master equation is a system of  $\binom{N}{L}$  coupled linear differential equations. It can be proved

that the distribution function  $P(n;t)$  finally becomes time independent. In the case of 'detailed balance' (i.e.  $W_{ji}[n^{ij}]P(n^{ij};t) = W_{ij}[n]P(n;t) \forall i, j, i \neq j$ ) the stationary distribution can be obtained. A less informative but often more tractable description is in terms of mean value equations (see page 16). The exact equations of motion for the mean values, which may be derived using (III), read:

$$\frac{d\bar{n}_j}{dt} = \sum_{i=1}^L \overline{W_{ji}[n;t]} - \sum_{i=1}^L \overline{W_{ij}[n;t]} \quad j=1, \dots, L \quad (IV)$$

where  $\overline{W_{ji}[n;t]} = \sum_n W_{ji}[n;t]P(n;t)$

Employing the approximation  $W_{ji}[n;t] = W_{ji}[\bar{n};t]$ , which is valid as long as the probability distribution remains narrow and unimodal (see footnote 1, page 17), we arrive at the self-contained set of coupled-differential equations:

$$\frac{d\bar{n}_j}{dt} = \sum_{i=1}^L W_{ji}[\bar{n};t] - \sum_{i=1}^L W_{ij}[\bar{n};t] \quad j=1, \dots, L \quad (V)$$

Since  $P_{ji}$  is a function of  $n$ , (V) is non-linear and may have distinct stationary points. For the relation between these mean value equations and the distribution function (see page 16-17).

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