A simplicial algorithm to solve the nonlinear complementarity problem on Sn x Rm+

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THE NONLINEAR COMPLEMENTARITY PROBLEM ON $S^n \times R^m_+$

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Abstract

In this paper a simplicial algorithm is developed to solve the nonlinear complementarity problem on $S^n \times R^m_+$. Furthermore, a condition for convergence is formulated. The triangulation which underlies the algorithm is a combination of the V-triangulation of S^n and the K-triangulation of R^m_+. Therefore we will call it the VK-triangulation.

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1. Introduction

Simplicial algorithms are nowadays frequently used to solve the zero point problem and the nonlinear complementarity problem on the n-dimensional unit simplex $S^n = \{x \in R^{n+1}_+ \mid \Sigma_i x_i = 1\}$. In economics simplicial algorithms have become a popular tool in computing equilibria of a pure exchange economy, as these equilibria can be formulated as nonlinear complementarity points of the so-called excess demand function.

Shortly, simplicial algorithms divide S^n into a finite number of simplices (this subdivision is called the triangulation of S^n) and search for a simplex which yields an approximate solution. Several triangulations of S^n have been developed which underly these algorithms. Van der Laan and Talman (1982) have made a first approach to generalize these triangulations and algorithms to the product space of N unit simplices S. Recently Doup (1987) has given a survey of possible triangulations and algorithms on S. This generalization makes the algorithms suitable for the computation of Nash equilibria in noncooperative N-person games and of equilibria in an economy with a block diagonal supply-demand pattern.

In Dirven and Talman (1987) a simplicial algorithm on $S^n \times R^m_+$ is developed to compute equilibria in economies with linear production. Now, S^n represents the space of commodity prices while R^m_+ represents the space of activity variables. A subdivision of $S^n \times R^m_+$ is developed which consists of (m+n)-dimensional cells, which are the carthesian product of n-simplices of the V-triangulation of S^n (see Doup and Talman (1987)) and of R^m_+. Because of the linearity of the problem in the activity variables no simplicial subdivision of R^m_+ is needed. However, there are economic applications, for example when the production exhibits increasing returns to scale, where the above described subdivision will not suffice. In these cases we will need a simplicial subdivision of $S^n \times R^a_+$.

In this paper we will develop a simplicial variable dimension restart algorithm which can be used to solve the nonlinear complementarity problem on $S^n \times R^m_+$. Furthermore, a convergence condition for this algorithm will be formulated. The triangulation which underlies the algorithm is a combination of the V-triangulation of S^n and the K-triangulation of R^m_+. Therefore we will call it the VK-triangulation.

A variable dimension restart algorithm is such that it generates a sequence of adjacent simplices of varying dimension, starting with a zero-dimensional simplex (the starting point which can be chosen arbitrarily) and ending within a finite number of steps with an approximating simplex, i.e. a simplex which yields an approximate solution. This sequence of adjacent simplices traces a piecewise linear path of points from the starting point to an approximate solution. The piecewise linear path is traced by alternating replacement steps in the simplicial subdivision in order to move from one simplex to an adjacent simplex and by linear programming pivot steps in a system of n+m+2 linear equations in order to trace a linear piece of the path in a given simplex.

This paper is organized as follows. In section 2 we will formulate the problem, section 3 discusses the triangulation of $S^n \times R^m_+$ which underlies the algorithm. Finally, in section 4 the algorithm will be exposed and theorems on convergence and accuracy will be given.

2. The problem

Let S^n be the n-dimensional unit simplex, i.e. $S^n = \{x \in R^{n+1}_+ \mid \Sigma_i x_i = 1\}$. Let v be a continuous function from $S^n \times R^m_+$ to $R^{n+1} \times R^m$. Furthermore, $v(p,y) = (z(p,y)^T, x(p,y)^T)^T$ with $p \in S^n$, $y \in R^m_+$, $z(p,y) \in R^{n+1}$, $x(p,y) \in R^m$ and v satisfies a condition equivalent to Walras' law: $(p^T,y^T)^T.v(p,y) = \Sigma_i p_i.z_i(p,y) + \Sigma_j y_j.x_j(p,y) = 0$ for all $p \in S^n$, $y \in R^m_+$.

The nonlinear complementarity problem on $S^n \times R^m_+$ is now to find to find a vector (p^*,y^*) such that $v(p^*,y^*) \leq 0$. Note that such a vector (p^*,y^*) has the property $z_i(p^*,y^*) = 0$ if $p^*_i > 0$ $(i=1,\ldots,n+1)$, $x_j(p^*,y^*) = 0$ if $y^*_j > 0$ $(j=1,\ldots,m)$, $z_i(p^*,y^*) \leq 0$ if $p^*_i = 0$ $(i=1,\ldots,n+1)$, $x_j(p^*,y^*) \leq 0$ if $y^*_j = 0$ $(j=1,\ldots,m)$. To solve the nonlinear complementarity problem on $S^n \times R^m_+$ we will develop in this paper a variable dimension restart algorithm which operates on $S^n \times R^m_+$.

Since $S^n \times R^m_+$ is not a compact set, we have to formulate some condition which guarantees convergence. As we will see in section 4, a sufficient condition for convergence is that $x_j(p,y) < -\psi$ (where ψ can be arbitrarily small) if $y_j > y_j^{max}$ for some finite y_j^{max}, $j=1,\ldots,m$.

Under this condition the algorithm is confined to a compact subset of $S^n \times R^m_+$ and convergence is assured.

The algorithm traces a piecewise linear path of points in a triangulation of $S^n \times R^m_+$, starting in $(\underline{p},\underline{y})$, such that for some α and β, $0 \leq \alpha \leq 1$, the following condition will be satisfied:

$$
\begin{aligned}
p_i &= \alpha.\underline{p}_i \text{ if } Z_i(p,y) < \beta \\
p_i &\geq \alpha.\underline{p}_i \text{ if } Z_i(p,y) = \beta \\
y_j &= \alpha.\underline{y}_j \text{ if } X_j(p,y) < \beta \\
y_j &\geq \alpha.\underline{y}_j \text{ if } X_j(p,y) = \beta
\end{aligned}
\tag{2.1}
$$

$Z(p,y)$ is the piecewise linear approximation to $z(p,y)$ given by $Z(p,y) = \Sigma_k\lambda_k.z(p^k,y^k)$ where (p,y) is a point in some t-simplex $\sigma(w^1,\ldots,w^{t+1})$ with vertices $w^k = (p^k,y^k)$, $k=1,\ldots,t+1$, in the triangulation of $S^n \times R^m_+$ and $(p,y) = \Sigma_k\lambda_k.(p^k,y^k)$. $X(p,y)$ is the piecewise linear approximation to $x(p,y)$ given by $X(p,y) = \Sigma_k\lambda_k.x(p^k,y^k)$. Note that $\beta = \max\{\max_i Z_i(p,y), \max_j X_j(p,y)\}$ and $\alpha = \min\{\min_i p_i/\underline{p}_i, \min_j y_j/\underline{y}_j\}$.

The algorithm will terminate with a point (p^*,y^*) such that:

$$
\begin{aligned}
Z_i(p^*,y^*) &= \beta \text{ if } p^*_i > 0 \\
Z_i(p^*,y^*) &\leq \beta \text{ if } p^*_i = 0 \\
X_j(p^*,y^*) &= \beta \text{ if } y^*_j > 0 \\
X_j(p^*,y^*) &\leq \beta \text{ if } y^*_j = 0
\end{aligned}
\tag{2.2}
$$

In section 4 it will be shown that such a point (p^*,y^*) yields an approximate nonlinear complementarity point of $v(p,y)$, which is the problem we want to solve.

3. The VK-triangulation of $S^n \times R^m_+$

In this section we will describe the triangulation which underlies the algorithm. Since the triangulation we use is a combination of the V-triangulation of S^n and the K-triangulation of R^m_+ we call it the VK-triangulation. The K-triangulation is constructed by Freudenthal (1942) and the V-triangulation is introduced in Doup and Talman (1987).

Let T_1 be a subset of I_{n+1} with $|T_1| = t_1$, T_2 a subset of $I_{n+m+1} \backslash I_{n+1}$ with $|T_2| = t_2$, $T = T_1 \cup T_2$ with $|T| = t$ and let $(\underline{p}, \underline{y})$ be some point in $S^n \times R^m_+$.

Definition 3.1

For $t \leq n+m$ we define $A(T_1 \cup T_2) = A(T)$ as

$$A(T_1 \cup T_2) = \{(p,y) \in S^n \times R^m_+ \mid \begin{array}{l} p_i = \alpha . \underline{p}_i \text{ if } i \notin T_1 \\ p_i \geq \alpha . \underline{p}_i \text{ if } i \in T_1 \\ y_j = \alpha . \underline{y}_j \text{ if } j+n+1 \notin T_2 \\ y_j \geq \alpha . \underline{y}_j \text{ if } j+n+1 \in T_2 \\ \qquad\qquad 0 \leq \alpha \leq 1 \;\} \end{array}$$

Analogously to the V-triangulation of S^n we divide a nonempty set $A(T)$ into $A(\gamma(T_1), T_2)$ for any permutation $\gamma(T_1) = (\gamma_1, \ldots, \gamma_{t1})$ of the t_1 elements of T_1. Let $q(\gamma_i)$ be defined as:

$$q(\gamma_1) = e(\gamma_1) - \begin{pmatrix} \underline{p} \\ \underline{y} \end{pmatrix}$$

$$q(\gamma_h) = \begin{pmatrix} P(\{\gamma_1, \ldots, \gamma_h\}) - P(\{\gamma_1, \ldots, \gamma_{h-1}\}) \\ 0 \end{pmatrix} \qquad h=2, \ldots, t_1$$

with $P(K)$ the relative projection in S^n of $\underline{p}$ on $S^n(K) = \{p \in S^n \mid p_j = 0,\ j \notin K\}$ (see definition 3.2 below) and $e(i)$ the i-th unit vector in R^{n+m+1}.

Definition 3.2

Let K be a nonempty subset of I_{n+1}, and let K^0 be the set given by $K^0 = \{i \in K \mid \underline{p}_i = 0\}$. The relative projection vector P(K) of $\underline{p}$ on $S^n(K)$ is given by:

$$P_h(K) = \begin{cases} 0 & h \notin K \\ \underline{p}_h(1+|K^0|)/(\Sigma_{k\in K}\, \underline{p}_k + |K^0|) & h \in K\backslash K^0 \\ (1 - \Sigma_{k\in K}\, \underline{p}_k)\backslash(\Sigma_{k\in K}\, \underline{p}_k + |K^0|) & h \in K^0 \end{cases}$$

For $K = \emptyset$ we define $P(\emptyset) = \underline{p}$.

Note that for $\underline{p}$ in the interior of S^n, in other words for $K^0 = \emptyset$, we have that $P_h(K) = \underline{p}_h/(\Sigma_{k\in K}\, \underline{p}_k)$ for $h \in K$ and $P_h(K) = 0$ for $h \notin K$.

Definition 3.3

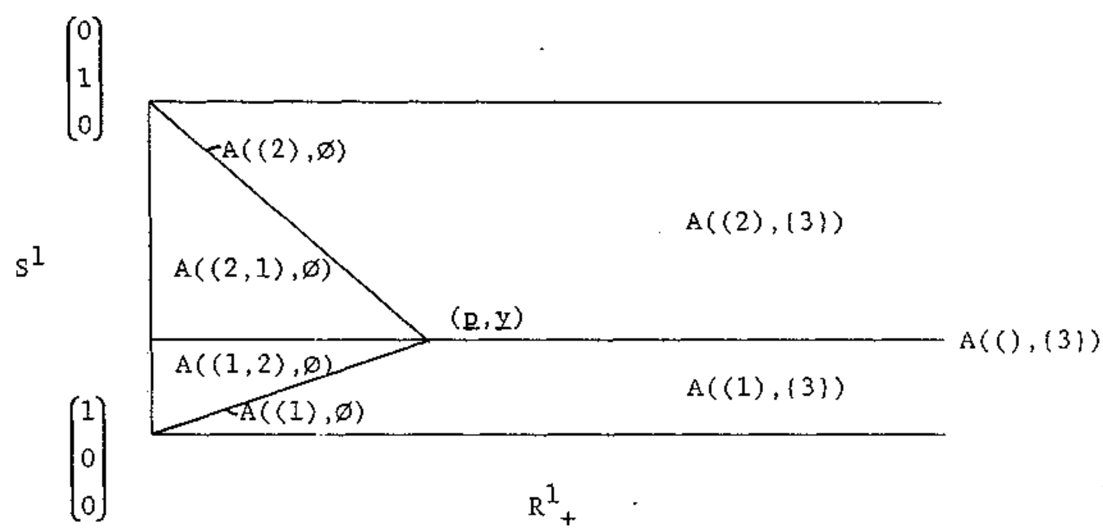
For $t_1+t_2 \le n+m$ we define $A(\gamma(T_1),T_2)$ as:

$$A(\gamma(T_1),T_2) = \{(p,y)\in S^n \times R^m_+ \mid (p,y) = (\underline{p},\underline{y}) + \sum_{i=1}^{t_1} \alpha_i \cdot q(\gamma_i) + \sum_{j\in T_2} \nu_j \cdot e(j)$$

$$0 \le \alpha_{t1} \le \ldots \le \alpha_1 \le 1,\ \nu_j \ge 0\}.$$

In figure 3.4 the sets $A(\gamma(T_1),T_2)$ are illustrated for n=m=1.

Figure 3.4 The sets $A(\gamma(T_1),T_2)$ for n=m=1



The union of $A(\gamma(T_1),T_2)$ over all permutations $\gamma(T_1)$ is $A(T_1 \cup T_2)$. The (t_1+t_2)-dimensional set $A(\gamma(T_1),T_2)$ is triangulated by the collection $G(\gamma(T_1),T_2)$ of (t_1+t_2)-simplices $\sigma(w^1,\pi(T))$ with vertices $w^1,\ldots,w^{t_1+t_2+1}$ such that:

(i) $$w^1 = \begin{pmatrix} \underline{p} \\ \underline{y} \end{pmatrix} + \sum_{i=1}^{t_1} a(\gamma_i)\, m^{-1}\, q(\gamma_i) + \sum_{j\in T_2} b_j\, m^{-1}\, \underline{y}_{j-(n+1)}\, e(j)$$

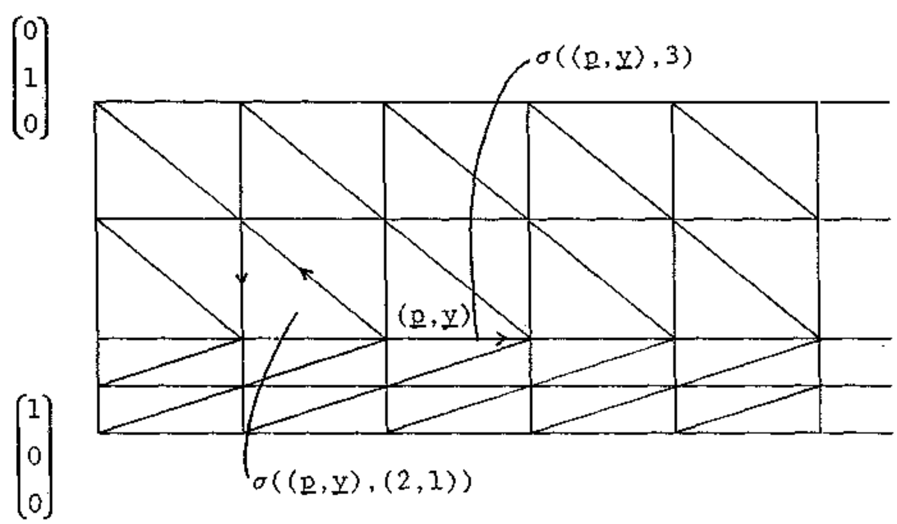
with $a(\gamma_i)$ and b_j integers and $0 \le a(\gamma_{t_1}) \le \ldots \le a(\gamma_1) \le m-1$, $b_j \ge 0$ and m the gridsize of the triangulation.

(ii) π is a permutation of the elements of T_1 and T_2
$\pi = (\pi_1,\ldots,\pi_{t_1+t_2})$.

(iii) $w^{k+1} = w^k + m^{-1}\, q(\pi_k)$
where $q(\pi_k) = q(\gamma_i)$ if $\pi_k = \gamma_i \in T_1$
and $q(\pi_k) = \underline{y}_{\pi_k-(n+1)} \cdot e(\pi_k)$ if $\pi_k \in T_2$.

The union $G(T_1 \cup T_2) = G(T)$ of the $G(\gamma(T_1),T_2)$'s over all permutation vectors γ of T_1 yields a triangulation of $A(T_1 \cup T_2)$ and the union of these triangulations $G(T)$ of $A(T)$ over all feasible T with $t \le n+m$ induces the VK-triangulation of $S^n \times R^m_+$ with gridsize m^{-1}. In figure 3.5 the triangulation is illustrated for $n=m=1$.

Figure 3.5 The VK-triangulation for n=m=1



Let $\sigma(w^1,\pi(T))$ and $\underline{\sigma}(\underline{w}^1,\underline{\pi}(T))$ be two adjacent simplices in $G(\gamma(T_1),T_2)$ with common facet τ opposite vertex w^r, $1 \le r \le t+1$, then $\underline{\sigma}$ is obtained from σ by the following table where the (n+m+1)-vector a is given by $a_i = a(i)$, $i \in T_1$, $a_i = b_i$, $i \in T_2$ and $a_i = 0$, $i \notin T$:

<u>Table 3.6</u> replacement of vertex w^r

	$\underline{w}^1$	$\underline{\pi}(T)$	$\underline{a}$
$r = 1$	$w^1 + m^{-1}.q(\pi_1)$	$(\pi_2,\ldots,\pi_t,\pi_1)$	$a + e(\pi_1)$
$1<r<t+1$	w^1	$(\pi_1,\ldots,\pi_{r-2},\pi_r,$ $\pi_{r-1},\pi_{r+1},\ldots,\pi_t)$	a
$r = t+1$	$w^1 - m^{-1}.q(\pi_t)$	$(\pi_t,\pi_1,\ldots,\pi_{t-1})$	$a - e(\pi_t)$

<u>Lemma 3.7</u>
Let $\sigma(w^1,\pi(T))$ be a simplex in $G(\gamma(T_1),T_2)$ and let τ be the facet of σ opposite vertex w^r, $1 \le r \le t+1$. Then τ lies in the boundary of $A(\gamma(T_1),T_2)$ if and only if one of the following cases holds:

(i) $r=1$, $\pi_1 = \gamma_1$, $a(\pi_1) = m-1$
(ii) $1 < r < t+1$, $\pi_r = \gamma_i$, $\pi_{r-1} = \gamma_{i-1}$, for some i, $2 \le i \le t_1$, $a(\pi_r) = a(\pi_{r-1})$
(iii) $r = t+1$, $\pi_t = \gamma_{t1}$, $a(\pi_t) = 0$
(iv) $r = t+1$, $\pi_t \in T_2$, $a(\pi_t) = 0$

Lemma 3.7 is a direct result of the definitions of $A(\gamma(T_1),T_2)$ and $G(\gamma(T_1),T_2)$. The following three lemma's describe the cases of lemma 3.7 in more detail. Let $S^n(T_1) \times R^m_+(T_2) = \{(p,y) \in S^n \times R^m_+ \mid p_i = 0$ for all $i \notin T_1$ and $y_{j-(n+1)} = 0$ for all $j \notin T_2\}$.

<u>Lemma 3.8</u>

Let $\sigma(w^1,\pi(T))$ be a t-simplex in $G(\gamma(T_1),T_2)$ with the facet τ of σ opposite vertex w^1 in the boundary of $A(\gamma(T_1),T_2)$, then τ lies in $S^n(T_1) \times R^m{}_+(T_2)$.

<u>Lemma 3.9</u>

Let $\sigma(w^1,\pi(T))$ be a t-simplex in $G(\gamma(T_1),T_2)$ with the facet τ of σ opposite vertex w^r, $1 < r < t+1$, in the boundary of $A(\gamma(T_1),T_2)$. Then, according to lemma 3.7(ii) there is some i such that $\pi_r=\gamma_i$, $\pi_{r-1}=\gamma_{i-1}$, $2 \le i \le t_1$, and $a(\pi_r) = a(\pi_{r-1})$. Now, τ is a facet of the t-simplex $\underline{\sigma}(w^1,\underline{\pi}(T))$ in $G(\underline{\gamma}(T_1),T_2)$ with $\underline{\gamma}(T_1)=(\gamma_1,\dots,\gamma_{i-2},\gamma_i,\gamma_{i-1},\gamma_{i+1},\dots,\gamma_{t1})$ and $\underline{\pi}(T) = (\pi_1,\dots,\pi_{r-2},\pi_r,\pi_{r-1},\pi_{r+1},\dots,\pi_t)$.

<u>Lemma 3.10</u>

Let $\sigma(w^1,\pi(T))$ be a t-simplex in $G(\gamma(T_1),T_2)$ with the facet τ of σ opposite vertex w^{t+1} in the boundary of $A(\gamma(T_1),T_2)$. If $\pi_t = \gamma_{t1}$ and $a(\pi_t) = 0$ then τ is the (t-1)-simplex $\underline{\sigma}(w^1,\pi(\underline{T}))$ in $G(\gamma(\underline{T}_1),T_2))$ with $\pi(\underline{T}) = (\pi_1,\dots,\pi_{t-1})$, $\underline{T}_1 = T_1\backslash\{\gamma_{t1}\}$ and $\gamma(\underline{T}_1) = (\gamma_1,\dots,\gamma_{t1-1})$. If $\pi_t \in T_2$ and $a(\pi_t) = 0$ then τ is the (t-1)-simplex $\underline{\sigma}(w^1,\pi(\underline{T}))$ in $G(\gamma(T_1),\underline{T}_2))$ with $\pi(\underline{T}) = (\pi_1,\dots,\pi_{t-1})$ and $\underline{T}_2 = T_2\backslash\{\pi_t\}$.

4. The algorithm

In this section we will describe the steps of the variable dimension restart algorithm on $S^n \times R^m_+$ and formulate conditions on convergence and accuracy. Let $S^n \times R^m_+$ be triangulated as described in section 3. To find an approximate solution to the nonlinear complementarity problem the algorithm traces, by alternating linear programming pivot steps and replacement steps, a piecewise linear path of points satisfying (2.1).

<u>Definition 4.1</u>

Let T_1 be a subset of I_{n+1} with $|T_1| = t_1$, T_2 a subset of $I_{n+m+1}\backslash I_{n+1}$ with $|T_2| = t_2$ and $T = T_1 \cup T_2$, $|T| = t = t_1+t_2$. A g-simplex $\sigma(w^1,\ldots,w^{g+1})$ with $g = t,t-1$ is <u>T-complete</u> if the system of linear equations:

$$\sum_{k=1}^{g+1} \lambda_k \begin{bmatrix} z(p^k,y^k) \\ x(p^k,y^k) \\ 1 \end{bmatrix} + \sum_{h\notin T} \mu_h \begin{bmatrix} e(h) \\ 0 \end{bmatrix} - \beta \begin{bmatrix} e \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{4.1}$$

(where e is a (n+m+1)-vector of ones), has a solution $\lambda_k \geq 0$, $k=1,\ldots,g+1$, $\mu_h \geq 0$, $h \notin T$ and β. Such a solution is called feasible and will be denoted by (λ,μ,β).

<u>Definition 4.2</u>

A T-complete (t-1)-simplex $\sigma(w^1,\ldots,w^t)$ is <u>complete</u> if for all (p,y) in σ: $p_i = 0$ if $i \notin T_1$ and $y_{j-(n+1)} = 0$ if $j \notin T_2$.

<u>Nondegeneracy Assumption</u>

If $\sigma(w^1,\ldots,w^{g+1})$ is a T-complete g-simplex in $S^n \times R^m_+$, then for g=t-1 system (4.1) has a unique solution (λ,μ,β) with $\lambda_k > 0$, $k=1,\ldots,t$, and $\mu_h > 0$, $h \notin T$, while for g=t at most one of the variables (λ,μ) is equal to zero at each feasible solution.

Under the nondegeneracy assumption a T-complete t-simplex σ contains a line segment of solutions (λ,μ,β) with $\lambda \geq 0$ and $\mu \geq 0$. A solution with one of the components of (λ,μ) equal to zero is called a basic solution. A line segment of solutions (λ,μ,β) can be traced by making a linear programming pivot step in (4.1). If λ_k is equal to zero at a basic solution the facet of σ opposite w^k is a T-complete (t-1)-simplex.

In order to trace the above mentioned piecewise linear path the algorithm generates for varying T a sequence of adjacent T-complete t-simplices with common T-complete facets in A(T). The algorithm starts with the unique 1-simplex $\sigma^0((\underline{p},\underline{y}),t^0)$ where t^0 is the (unique) index such that $v_{t0}(\underline{p},\underline{y}) = \max_h v_h(\underline{p},\underline{y})$. σ^0 is $\{t^0\}$-complete with a basic solution equal to $\lambda_1 = 1$, $\lambda_2 = 0$, $\mu_h = v_{t0}(\underline{p},\underline{y}) - v_h(\underline{p},\underline{y})$, $h \neq t^0$, and $\beta = v_{t0}(\underline{p},\underline{y})$. Now, a linear programming pivot step is made with $(v(p^2,y^2),1)$ in the linear system (4.1) corresponding with σ^0 by increasing λ_2 away from zero where $(p^2,y^2) = (\underline{p},\underline{y}) + m^{-1}.q(t^0)$ if $t^0 \in I_{n+1}$ and $(p^2,y^2) = (\underline{p},\underline{y}) + m^{-1}.\underline{y}_{t0-(n+1)}.e(t^0)$ if $t^0 \in I_{n+m+1}\backslash I_{n+1}$.

For given T_1, $\gamma(T_1)$ and T_2 the T-complete t-simplices $\sigma(w^1,\pi(T))$ in $G(\gamma(T_1),T_2)$ form sequences of adjacent simplices with T-complete common facets. Since each T-complete t-simplex in $G(\gamma(T_1),T_2)$ has at most two T-complete facets and each facet of a t-simplex is either a facet of another t-simplex in $G(\gamma(T_1),T_2)$ or is a facet in the boundary of $A(\gamma(T_1),T_2)$, these sequences are either loops or have two end simplices. An end simplex is either a t-simplex with a T-complete facet τ in the boundary of $A(\gamma(T_1),T_2)$ (this corresponds with one of the cases of lemma 3.7) or a T-complete t-simplex having a solution (λ,μ,β) with $\mu_s = 0$ for some $s \notin T$. In the latter case σ is also $T\cup\{s\}$-complete. The $T\cup\{s\}$-complete t-simplex σ is complete if σ lies in $S^n(T_1\cup\{s\}) \times R^m_+(T_2)$ if $s \in I_{n+1}$ and if σ lies in $S^n(T_1) \times R^m_+(T_2\cup\{s\})$ if $s \in I_{n+m+1}\backslash I_{n+1}$. Note that σ is always complete if $|T| = n+m$, since then $S^n(T_1\cup\{s\}) \times R^m_+(T_2)$ respectively $S^n(T_1) \times R^m_+(T_2\cup\{s\})$ equals $S^n \times R^m_+$. If σ is not complete then σ is a facet of just one (t+1)-simplex $\underline{\sigma}$ in $G((\gamma(T_1),s),T_2)$ respectively in $G(\gamma(T_1),T_2\cup\{s\})$ as described in lemma 4.3 below. In the former case, i.e in the case that an end simplex has a T-complete facet τ in the boundary of $A(\gamma(T_1),T_2)$, we have, if the T-complete facet τ lies in $S^n(T_1) \times R^m_+(T_2)$ then τ is a complete simplex (see lemma 3.8), if the facet τ does not lie in $S^n(T_1) \times R^m_+(T_2)$ then τ is either a facet of a T-complete t-simplex in $G(\underline{\gamma}(T_1),T_2)$ as described in lemma 3.9 or τ is a T-complete (t-1)-simplex in $G(\gamma(\underline{T}_1),T_2)$ or in $G(\gamma(T_1),\underline{T}_2)$ as described in lemma 3.10.

Lemma 4.3

Let $\sigma(w^1,\pi(T))$ be a $T\cup\{s\}$-complete simplex in $G(\gamma(T_1),T_2)$. If σ is not complete then σ is a facet of the $\underline{T}$-complete (t+1)-simplex $\underline{\sigma}(w^1,\pi(\underline{T}))$ with $\underline{T} = T\cup\{s\}$ and $\pi(\underline{T}) = (\pi_1,\ldots,\pi_{t+1},s)$. If $\{s\} \in I_{n+1}$ then $\underline{\sigma}$ lies in $A(\gamma(\underline{T}_1),T_2)$ with $\gamma(\underline{T}_1) = (\gamma_1,\ldots,\gamma_{t1},s)$. If $\{s\} \in I_{n+m+1}\backslash I_{n+1}$ then $\underline{\sigma}$ lies in $A(\gamma(T_1),\underline{T}_2)$ with $\underline{T}_2 = T_2 \cup \{s\}$.

From the above discussion we can conclude that for varying T the sequences of T-complete simplices in G(T) can be linked to form sequences of adjacent simplices of varying dimension. Under the convergence assumption as formulated below the algorithm is confined to a compact subset of $S^n \times R^m_+$ (see theorem 4.5) and consequently, a sequence is either a loop or a path having two end simplices. One end simplex is the 1-dimensional simplex $\sigma^0((\underline{p},\underline{y}),t^0)$, whereas all other end simplices are complete simplices. The path starting in σ^0 connects the starting point $(\underline{p},\underline{y})$ with a complete simplex. This path is generated by the algorithm. In theorem 4.6 it is shown that a complete simplex yields an approximate solution to the nonlinear complementarity problem on $S^n \times R^m_+$.

In the following lemma and theorems v is a continuous function from $S^n \times R^m_+$ to $R^{n+1} \times R^m$ as given in section 2. So, $v(p,y) = (z(p,y)^T,x(p,y)^T)^T$ with $p \in S^n$, $y \in R^m_+$, $z(p,y) \in R^{n+1}$, $x(p,y) \in R^m$ and v satisfies a condition equivalent to Walras' law, i.e. $(p^T,y^T)^T.v(p,y) = \Sigma_i p_i.z_i(p,y) + \Sigma_j y_j.x_j(p,y) = 0$ for all $p \in S^n$, $y \in R^m_+$. Furthermore, Z(p,y) respectively X(p,y) is the piecewise linear approximation to z(p,y) respectively x(p,y) with respect to the chosen VK-triangulation of $S^n \times R^m_+$.

Lemma 4.4

For all $\epsilon > 0$, for all $\chi > 0$, there is a $\delta(\epsilon,\chi) > 0$ such that for each VK-triangulation with mesh size $\delta' < \delta(\epsilon,\chi)$, for all (p,y), $p \in S^n$, $y_j \in [0,y_j^{max}+2\chi]$, for given $y_j^{max} > 0$, j=1,...,m, holds that:

$$|p^T.Z(p,y) + y^T.X(p,y)| < (1+\Sigma_j y_j).\epsilon$$

Proof

Since v is a continuous function and $S^n \times [0,y_1^{max}+2\chi] \times \ldots \times [0,y_m^{max}+2\chi]$ is a compact set there is a $\delta > 0$ such that for all $r,s \in S^n \times [0,y_1^{max}+2\chi] \times \ldots \times [0,y_m^{max}+2\chi]$ holds that $\max_i |r_i - s_i| < \delta$ implies $\max_i |v_i(r) - v_i(s)| < \epsilon$. Consequently, for $(p,y) \in \sigma(w^1,\ldots,w^{t+1})$ with $w^k = (p^k,y^k)$, $k=1,\ldots,t+1$, $(p,y) = \Sigma_k\lambda_k(p^k,y^k)$, $\lambda_k \geq 0$, $\Sigma\lambda_k = 1$, we have, using "Walras' law":

$$|p^T.Z(p,y) + y^T.X(p,y)| =$$

$$|\Sigma_i p_i Z_i(p,y) + \Sigma_j y_j X_j(p,y) - (\Sigma_i p_i.z_i(p,y) + \Sigma_j y_j.x_j(p,y))| =$$

$$|\Sigma_i p_i(Z_i(p,y) - z_i(p,y)) + \Sigma_j y_j(X_j(p,y) - x_j(p,y))| =$$

$$|\Sigma_i p_i(\Sigma_k\lambda_k(z_i(p^k,y^k) - z_i(p,y))) + \Sigma_j y_j(\Sigma_k\lambda_k(x_j(p^k,y^k) - x_j(p,y))| <$$

$$(1+\Sigma_j y_j).\epsilon,$$

since $|z_i(p^k,y^k) - z_i(p,y)| < \epsilon$ and $|x_j(p^k,y^k) - x_j(p,y)| < \epsilon$.

□

Convergence Assumption

There are finite $y_j^{max} > 0$, $j=1,\ldots,m$ such that $x_j(p^k,y^k) < -\psi$, for some $\psi > 0$, if $y^k_j > y_j^{max}$, $j=1,\ldots,m$.

Now, using lemma 4.4 with $\epsilon = \psi$ we can show that under the convergence assumption the algorithm is confined to a compact set.

Theorem 4.5

Let $\chi > 0$. Under the convergence assumption there is a mesh size $\delta' > 0$ such that the algorithm, when operating in a triangulation of $S^n \times R^m_+$ with mesh size δ', is confined to the compact set $S^n \times [0,y_1^{max}+\chi] \times \ldots \times [0,y_m^{max}+\chi]$.

Proof

Let $\delta' < \min(\delta(\psi,\chi),\chi)$. Suppose there is a t-simplex σ' with vertices $w'^k = (p'^k,y'^k)$, $k=1,\ldots,t+1$, which is visited by the algorithm, such that for some h, for some k, $y'^k_h > y_h^{max}+\chi$. Note that σ' does not lie in $S^n \times [0,y_1^{max}+\chi] \times \ldots \times [0,y_m^{max}+\chi]$. If σ' is visited by the algorithm there must be some t-simplex σ with vertices $w^k = (p^k,y^k)$, $k=1,\ldots,t+1$, be visited by the algorithm with $y^k_h \in (y_h^{max},y_h^{max}+2\chi]$ for all k. But then, according to the definition of $A(\gamma(T_1),T_2)$ σ must lie in some $A(\gamma(T_1),T_2)$ with T_2 such that $h+n+1 \in T_2$ and σ must be T-complete. According to the convergence assumption, $x_h(p^k,y^k) < -\psi$ for all k and consequently, $X_h(p,y) = \Sigma_k\lambda_k x_h(p^k,y^k) < -\psi$. Furthermore, since σ is T-complete we have according to (4.1) that $X_h(p,y) = \max(\max_i Z_i(p,y), \max_j X_j(p,y))$ and therefore $Z_i(p,y) < -\psi$, $i=1,\ldots,n+1$ and $X_j(p,y) < -\psi$, $j=1,\ldots,m$. But then we have $p.Z(p,y) + y.X(p,y) < -(1+\Sigma_j y_j).\psi$. However, according to theorem 4.4 this cannot be the case. In other words such a simplex σ cannot be visited by the algorithm. Consequently, also σ' cannot be visited by the algorithm and the algorithm is confined to $S^n \times [0,y_1^{max}+\chi] \times \ldots \times [0,y_m^{max}+\chi]$.

□

Theorem 4.6

For all $\epsilon > 0$, for all $\chi > 0$ there is a mesh size $\delta(\epsilon,\chi) > 0$ such that each complete simplex in $S^n \times [0,y_1^{max}+\chi] \times \ldots \times [0,y_m^{max}+\chi]$ with solution $(\lambda^*,\mu^*,\beta^*)$ to (4.1) in the VK-triangulation with mesh size smaller than $\delta(\epsilon,\chi)$ contains a point (p^*,y^*) with:

$$|\beta^*| < \epsilon$$

$$\beta^* - \epsilon < z_i(p^*,y^*) < \beta^* + \epsilon \qquad \text{if } p^*_i > 0$$

$$z_i(p^*,y^*) < \beta^* + \epsilon \qquad \text{if } p^*_i = 0$$

$$\beta^* - \epsilon < x_j(p^*,y^*) < \beta^* + \epsilon \qquad \text{if } y^*_j > 0$$

$$x_j(p^*,y^*) < \beta^* + \epsilon \qquad \text{if } y^*_j = 0$$

Proof

Let $\sigma^*(w^1,\ldots,w^t)$ be a complete simplex in the VK-triangulation with mesh size smaller than $\delta(\epsilon,\chi)$. The linear system (4.1) with respect to σ^* has a solution $(\lambda^*,\mu^*,\beta^*)$ such that ${\lambda^*}_k > 0$, $k=1,\ldots,t$ and ${\mu^*}_h > 0$, $h \notin T$. Let (p^*,y^*) be given by $(p^*,y^*) = \Sigma_k {\lambda^*}_k w^k$. Since $\Sigma_k {\lambda^*}_k = 1$, (p^*,y^*) lies in σ^*. Furthermore, we have from (4.1) that $\Sigma_k {\lambda^*}_k z_i(p^k,y^k) = \beta^*$ if $i \in T_1$, $\Sigma_k {\lambda^*}_k z_i(p^k,y^k) = \beta^* - {\mu^*}_i$ if $i \notin T_1$, $\Sigma_k {\lambda^*}_k x_j(p^k,y^k) = \beta^*$ if $j \in T_2$, $\Sigma_k {\lambda^*}_k x_j(p^k,y^k) = \beta^* - {\mu^*}_j$ if $j \notin T_2$. Consequently, since $(p^{*T},y^{*T}).v(p^*,y^*) = 0$ we have:

$$|(1+\Sigma {y^*}_j)\beta^*|$$

$$|(1+\Sigma {y^*}_j)\beta^* - (p^{*T},y^{*T}).v(p^*,y^*)| =$$

$$|\beta^* - p^{*T}.z(p^*,y^*) + \Sigma {y^*}_j.\beta^* - y^{*T}.x(p^*,y^*)| \le$$

$$|\beta^* - p^{*T}.z(p^*,y^*)| + |\Sigma {y^*}_j.\beta^* - y^{*T}.x(p^*,y^*)| =$$

$$|p^{*T}(\Sigma_k {\lambda^*}_k z(p^k,y^k) - z(p^*,y^*))| + |y^{*T}(\Sigma_k {\lambda^*}_k x(p^k,y^k) - x(p^*,y^*))| <$$

$$\epsilon + \Sigma {y^*}_j.\epsilon = (1+\Sigma {y^*}_j)\epsilon,$$

since ${p^*}_i = 0$ if $i \notin T_1$ and ${y^*}_j = 0$ if $j \notin T_2$,

$|z_i(p^k,y^k) - z_i(p^*,y^*)| < \epsilon$ and $|x_j(p^k,y^k) - x_j(p^*,y^*)| < \epsilon$.

So, $|\beta^*| < \epsilon$.

Furthermore,

$$|z_i(p^*,y^*) - \beta^*| = |\Sigma_k {\lambda^*}_k(z_i(p^*,y^*) - z_i(p^k,y^k))| < \epsilon \qquad \text{if } {p^*}_i > 0$$

$$z_i(p^*,y^*) - \beta^* \le \Sigma_k {\lambda^*}_k(z_i(p^*,y^*) - z_i(p^k,y^k)) < \epsilon \qquad \text{if } {p^*}_i = 0$$

$$|x_j(p^*,y^*) - \beta^*| = |\Sigma_k {\lambda^*}_k(x_j(p^*,y^*) - x_j(p^k,y^k))| < \epsilon \qquad \text{if } {y^*}_j > 0$$

$$x_j(p^*,y^*) - \beta^* \le \Sigma_k {\lambda^*}_k(x_j(p^*,y^*) - x_j(p^k,y^k)) < \epsilon \qquad \text{if } {y^*}_j = 0$$

□

From the above theorem we can conclude that the point (p^*,y^*) in a complete simplex σ^* gives an approximate complementarity point on $S^n \times [0,y_1^{max}+\chi] \times \ldots \times [0,y_m^{max}+\chi]$ with accuracy bounds given by 2ϵ. In most economic applications we can indeed restrict the problem to $S^n \times [0,y_1^{max}+\chi] \times \ldots \times [0,y_m^{max}+\chi]$. Combining theorem 4.5 and 4.6 we derive that when the algorithm operates in the VK-triangulation with mesh size $\delta' < \min(\delta(\epsilon,\chi),\delta(\psi,\chi),\chi)$ both convergence and an accuracy of 2ϵ is assured.

Finally, we give the steps of the algorithm in detail:

Step 0

Let t^0 be the (unique) index such that $v_{t0}(\underline{p},\underline{y}) = \max_h v_h(\underline{p},\underline{y})$. If $(\underline{p},\underline{y}) = e(t^0)$ then $\{(\underline{p},\underline{y})\}$ is complete and the algorithm terminates. Otherwise set $T = \{t^0\}$ and $t = 1$. Furthermore, set $T_1 = t^0$ and $T_2 = \emptyset$ if $t^0 \in I_{n+1}$ and $T_1 = \emptyset$ and $T_2 = t^0$ if $t^0 \in I_{n+m+1}\backslash I_{n+1}$. Set $w^1 = (\underline{p},\underline{y})$, $\pi(T) = (t^0)$, $\sigma = \sigma(w^1,\pi(T))$, $\underline{r} = 2$, $a_i = 0$, $i \in I_{n+m+1}$, $\lambda_1 = 1$, $\mu_h = v_{t0}(\underline{p},\underline{y}) - v_h(\underline{p},\underline{y})$ $h \neq t^0$ and $\beta = v_{t0}(\underline{p},\underline{y})$.

Step 1

Perform a linear programming pivot step by bringing $(z(w^{\underline{r}}),x(w^{\underline{r}}),1)$ in the linear system:

$$\sum_{\substack{k=1\\k\neq\underline{r}}}^{t+1} \lambda_k \begin{pmatrix} z(p^k,y^k) \\ x(p^k,y^k) \\ 1 \end{pmatrix} + \sum_{h\notin T} \mu_h \begin{pmatrix} e(h) \\ 0 \end{pmatrix} - \beta \begin{pmatrix} e \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

If μ_s becomes zero for some $s \notin T$ then goto step 3. Otherwise λ_r becomes zero for some $r \neq \underline{r}$.

Step 2

If $r = 1$, $\pi_1 = \gamma_1$ and $a(\pi_1) = m-1$ then the facet of σ opposite the vertex w^1 is a complete simplex and the algorithm terminates.

If $1 < r < t+1$, $\pi_r = \gamma_i$, $\pi_{r-1} = \gamma_{i-1}$, $2 \le i \le t_1$ and $a(\pi_r) = a(\pi_{r-1})$ then $\sigma(w^1,\pi(T))$ and $\gamma(T_1)$ are adapted as given in lemma 3.9. Return to step 1 with $\underline{r} = r$.

If $r = t+1$, $\pi_t = \gamma_{t1}$ and $a(\pi_t) = 0$ then set $s = \pi_t$, $T_1 = T_1 \backslash \{s\}$ and $t = t-1$, while $\sigma(w^1,\pi(T))$ and $\gamma(T_1)$ are adapted according to lemma 3.10. Go to step 4.

If $r = t+1$, $\pi_t \in T_2$ and $a(\pi_t) = 0$ then set $s = \pi_t$, $T_2 = T_2 \backslash \{s\}$ and $t = t-1$, while $\sigma(w^1,\pi(T))$ is adapted according to lemma 3.10. Go to step 4.

In all other cases $\sigma(w^1,\pi(T))$ and a are adapted according to table 3.6 by replacing w^r. Return to step 1 with $\underline{r}$ the index of the new vertex of σ.

Step 3

If $s \in I_{n+1}$, then if $p_i = 0$ for all $i \notin T_1 \cup \{s\}$ and $y_{j-(n+1)} = 0$ for all $j \notin T_2$ then σ is a complete simplex and the algorithm terminates. Otherwise set $T_1 = T_1 \cup \{s\}$ and $t = t+1$, while $\sigma(w^1,\pi(T))$ and $\gamma(T_1)$ are adapted according to lemma 4.3. Return to step 1 with $\underline{r} = t+1$.

If $s \in I_{n+m+1} \backslash I_{n+1}$, then if $p_i = 0$ for all $i \notin T_1$ and $y_{j-(n+1)} = 0$ for all $j \notin T_2 \cup \{s\}$ then σ is a complete simplex and the algorithm terminates. Otherwise set $T_2 = T_2 \cup \{s\}$ and $t = t+1$, while $\sigma(w^1,\pi(T))$ is adapted according to lemma 4.3. Return to step 1 with $\underline{r} = t+1$.

Step 4

Perform a linear programming pivot step by bringing $(e(s),0)$ in the linear system:

$$\sum_{k=1}^{t+1} \lambda_k \begin{pmatrix} z(p^k,y^k) \\ x(p^k,y^k) \\ 1 \end{pmatrix} + \sum_{\substack{h \notin T \\ h \neq s}} \mu_h \begin{pmatrix} e(h) \\ 0 \end{pmatrix} - \beta \begin{pmatrix} e \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

If μ_h becomes zero for some $h \notin T$, $h \neq s$, then return to step 3. Otherwise, λ_r becomes zero for some r and return to step 2.

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