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A NOTE ON THE LIMITING DISTRIBUTION OF SAMPLE AUTOCORRELATIONS IN THE PRESENCE OF A UNIT ROOT

by

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A M S T E R D A M

A NOTE ON THE LIMITING DISTRIBUTION OF SAMPLE AUTOCORRELATIONS IN THE PRESENCE OF A UNIT ROOT

by

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1. INTRODUCTION

It is well-known that the autocorrelation function of a stationary time series tails off to zero. This property is the basis for the well-known Box-Jenkins approach to determine the order of integration of a time series process: If the sample autocorrelations stay close to one then this indicates the presence of a unit root. The more modern approaches to testing the unit root hypothesis are based on the OLS estimator of the AR parameter in a first-order autoregression. See Fuller (1976), Dickey and Fuller (1979,1981), Evans and Savin (1981, 1984), Said and Dickey (1984), Dickey, Hasza and Fuller (1984), Phillips (1987), Phillips and Perron (1988), Hylleberg and Mizon (1989), Haldrup and Hylleberg (1989) and Pantula (1989), among others, for various unit root tests and Schwert (1989) for a Monte Carlo analysis of the power of some of these tests.

In practice, however, it sometimes happens that unit root tests do not reject the null hypothesis of a unit root although the sample auto-

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correlations tail off rather quickly. An empirical example of this phenomenon is the inflation rate in the US, based on the monthly price index of finished goods for the period 1960:01,..,1987:12 [source: OECD]. For this series the Phillips (1987) and Phillips-Perron (1988) test statistics Z_α take the values -2.683 and -5.375, respectively, where the Newey-West (1987) type variance estimator used in the construction of Z_α has been calculated with lag width $\ell = [n^{0.2}]$. Since these values are larger than the 10% critical values -5.7 and -11.3, respectively [cf. Fuller (1976, p.371)], one cannot reject the unit root hypothesis at the 10% significance level. On the other hand, the sample autocorrelations of the US infation rate in Table 1 below tail off rather quickly, which seems to contradict the unit root hypothesis. In order to explain this result, we shall derive the limiting distribution of the sample autocorrelations under the unit root hypothesis.

Table 1: Sample Autocorrelations $r_n(m)$ for the US Inflation Rate (Monthly data; 324 observations)

m	$r_n(m)$	m	$r_n(m)$	m	$r_n(m)$	m	$r_n(m)$	m	$r_n(m)$	m	$r_n(m)$
1	0.99	7	0.87	13	0.69	19	0.50	25	0.33	31	0.23
2	0.97	8	0.84	14	0.66	20	0.47	26	0.31	32	0.21
3	0.95	9	0.82	15	0.63	21	0.44	27	0.29	33	0.20
4	0.93	10	0.78	16	0.60	22	0.41	28	0.27	34	0.19
5	0.92	11	0.75	17	0.57	23	0.38	29	0.25	35	0.19
6	0.89	12	0.71	18	0.53	24	0.35	30	0.24	36	0.18

2. THE LIMITING DISTRIBUTION OF SAMPLE AUTOCORRELATIONS.

We recall that the sample autocorrelation function $r_n(m)$ takes the form:

$$(2.1)\quad r_n(m) = \frac{(1/(n-m))\Sigma_{t=m+1}^{n}(y_t-\bar{y})(y_{t-m}-\bar{y})}{\{(1/(n-m))\Sigma_{t=m+1}^{n}(y_{t-m}-\bar{y})^2\}^{\frac{1}{2}}\{(1/(n-m))\Sigma_{t=m+1}^{n}(y_t-\bar{y})^2\}^{\frac{1}{2}}},$$

where $m \geq 1$ and $\bar{y} = (1/n)\Sigma_{t=1}^{n}y_t$. It will be shown that $(n/m)(r_n(m)-1)$ converges in distribution to a function of a standard Wiener process, under similar assumptions on $u_t = y_t-y_{t-1}$ as in Phillips (1987):

ASSUMPTION 1: $u_t = y_t - y_{t-1}$; $E(u_t) = 0$ *for all* t; $\sup_t E|u_t|^\beta < \infty$ *for some* $\beta > 4$; $\sigma^2 = \lim_{n\to\infty} E[(1/\sqrt{n})\Sigma_{t=1}^n u_t]^2$ *exists and* $\sigma^2 > 0$; $\{u_t\}_1^\infty$ *is* α-*mixing with mixing coefficients* $\alpha(s)$ *that satisfy* $\Sigma_{s=1}^\infty \alpha(s)^{1-4/\beta} < \infty$.

Remark: Phillips (1987) assumes $\beta > 2$ and $\Sigma_{s=1}^\infty \alpha(s)^{1-2/\beta} < \infty$. Thus, Assumption 1 implies Assumption 2.1 of Phillips (1987), and consequently the results of Phillips (1987) and Phillips and Perron (1988) hold true under Assumption 1.

Let us introduce some notation and basic results. First, we denote by W a standard Wiener process [cf. Billingsley (1968)], that is, W is a stochastic element of the metric space $\{C[0,1],\rho\}$ of continuous real functions on [0,1], endowed with the norm $\rho(g,f) = \sup_{0\le r\le 1}|f(r)-g(r)|$, with the following properties: for all $r,s \in [0,1]$ with $r < s$, $W(r) \sim N(0,r)$, $W(s-r) \sim N(0,s-r)$, $W(r)$ and $W(s-r)$ are independent. Moreover, let

(2.2) $W_n(r) = \Sigma_{t=1}^{[rn]} u_t/(\sigma\sqrt{n})$ for $r \in [1/n,1]$; $W_n(r) = 0$ for $r \in [0,1/n)$.

This is a stochastic element of the metric space $\{D[0,1],\rho\}$ of real functions on [0,1] with countably many discontinuities, again endowed with the "sup" norm ρ. Phillips (1987) proves that under Assumption 1,

(2.3) $W_n \Rightarrow W$,

where "$\Rightarrow$" means "converges weakly to" [cf. Billingsley (1968)]. Then by the continuous mapping theorem [cf. Billingsley (1968), p.30],

(2.4) $\Phi(W_n) \Rightarrow \Phi(W)$, for any continuous mapping Φ of $\{D[0,1],\rho\}$ onto $\{D[0,1],\rho\}$, $\{C[0,1],\rho\}$ or R^k.

In the case that Φ maps $\{D[0,1],\rho\}$ onto R^k, $\Phi(W_n)$ and $\Phi(W)$ are random vectors, hence (2.4) then reads as: $\Phi(W_n) \to \Phi(W)$ in distribution.

Furthermore, let

(2.5) $\sigma^2(m) = \lim_{n\to\infty}(1/n)\Sigma_{t=m+1}^n E[(1/\sqrt{m})\Sigma_{j=t-m+1}^t u_j]^2$,

and observe that by Assumption 1,

(2.6) $\lim_{m\to\infty}\sigma^2(m) = \sigma^2$.

Finally, denote

(2.7) $$d_n(m) = \frac{\{\Sigma_{t=m+1}^n(y_{t-m}-\overline{y})^2\}^{1/2}}{\{\Sigma_{t=m+1}^n(y_t-\overline{y})^2\}^{1/2}}.$$

In the Appendix we shall prove:

LEMMA 1. *Let Assumption 1 hold and let* m *be a natural number, possibly converging to infinity with* n *at rate* o(n). *Then:*

(a) $\overline{y}/\sqrt{n} \Rightarrow \sigma\int_0^1 W(r)dr$;

(b) $(1/n^2)\Sigma_{t=m+1}^n(y_{t-m}-\overline{y})^2 \Rightarrow \sigma^2(\int_0^1 W(r)^2dr - [\int_0^1 W(r)dr]^2)$;

(c) $(1/n^2)\Sigma_{t=m+1}^n(y_t-\overline{y})^2 \Rightarrow \sigma^2(\int_0^1 W(r)^2dr - [\int_0^1 W(r)dr]^2)$;

(d) $(1/(mn))\Sigma_{t=m+1}^n(y_t-y_{t-m})y_{t-m} \Rightarrow \frac{1}{2}\sigma^2[W(1)^2-\sigma^2(m)/\sigma^2]$ *if* m *is fixed,*

$\Rightarrow \frac{1}{2}\sigma^2[W(1)^2-1]$ if $m \to \infty$ *at rate* $o(n^{1/3})$;

(e) $(1/(m\sqrt{n}))\Sigma_{t=m+1}^n(y_t-y_{t-m}) \Rightarrow \sigma W(1)$;

(f) $$(n/m)(d_n(m)-1) \Rightarrow \frac{-\frac{1}{2}W(1)^2 + W(1)\int_0^1 W(r)dr}{\int_0^1 W(r)^2dr - [\int_0^1 W(r)dr]^2};$$

(g) The above results hold simultaneously, i.e., stacking the random

variables at the left-hand side of "⇒" into a vector Z_n *and the random variables at the right-hand side into a vector* Z, *we have* $Z_n \Rightarrow Z$.

Before we derive the limiting distribution of $(n/m)(r_n(m)-1)$, we first consider the limiting distribution of the following variant of the sample autocorrelation function:

$$(2.8)\quad r_n^*(m) = \frac{(1/(n-m))\Sigma_{t=m+1}^{n}(y_t-\bar{y})(y_{t-m}-\bar{y})}{(1/(n-m))\Sigma_{t=m+1}^{n}(y_{t-m}-\bar{y})^2}.$$

Observe that

$$(2.9)\quad r_n^*(m) - 1 = \frac{\Sigma_{t=m+1}^{n}(y_t-y_{t-m})y_{t-m}}{\Sigma_{t=m+1}^{n}(y_{t-m}-\bar{y})^2} - \frac{\Sigma_{t=m+1}^{n}(y_t-y_{t-m})\bar{y}}{\Sigma_{t=m+1}^{n}(y_{t-m}-\bar{y})^2}.$$

Applying Lemma 1 to the numerators and denumerators in (2.9), it is easy to verify that the following lemma holds.

LEMMA 2: *Let Assumption 1 hold and let* $X_n^*(m) = (n/m)(r_n^*(m)-1)$,

$$X^*(m) = \frac{\frac{1}{2}[W(1)^2-\sigma^2(m)/\sigma^2]-W(1)\int_0^1 W(r)dr}{\int_0^1 W(r)^2 dr-[\int_0^1 W(r)dr]^2} \quad \textit{and}$$

$$X^* = \frac{\frac{1}{2}[W(1)^2-1]-W(1)\int_0^1 W(r)dr}{\int_0^1 W(r)^2 dr-[\int_0^1 W(r)dr]^2}.$$

(a) *For any fixed natural number* m, $X_n^*(m) \Rightarrow X^*(m)$.

(b) *If* $m \to \infty$ *at rate* $o(n^{1/3})$, *then* $X_n^*(m) \Rightarrow X^*$.

Note that the distribution of X^* is the same as the asymptotic distribution of the Phillips-Perron (1988) test Z_α, which in its turn is just the asymptotic distribution of $n(\tau_n-1)$, where now τ_n is the OLS estimate of the parameter τ in the regression model $y_t = c + \tau y_{t-1} + e_t$, $t=1,..,n$, with $e_t \sim NID(0,\sigma^2)$, $y_t = 0$ for $t \le 0$, $\tau = 1$ and $c = 0$. The distribution of X^* is tabulated in Fuller (1976, Table 8.5.1, p.371).

Next, observe that

$$(2.10)\quad r_n(m) - 1 = (r_n^*(m)-1)d_n(m) + d_n(m) - 1.$$

Combining (2.10) and the results of Lemma 2 and 3 with parts (f) and (g) of Lemma 1, our main result follows:

THEOREM 1: *Let Assumption 1 hold and let* $X_n(m) = (n/m)(r_n(m)-1)$, $X = -\frac{1}{2}/(\int_0^1 W(r)^2dr-[\int_0^1 W(r)dr]^2)$ *and* $c(m) = \sigma^2(m)/\sigma^2$.

(a) For any fixed natural number m, $X_n(m) \Rightarrow c(m)X$.

(b) If $m \to \infty$ *at rate* $o(n^{1/3})$, *then* $X_n(m) \Rightarrow X$.

To the best of our knowledge, the distribution of X has not yet been tabulated in the literature. Therefore, we have estimated the distribution of X by Monte Carlo simulation, using 10,000 replications of the random variable $-\frac{1}{2}/[(1/n^2)\Sigma_{t=1}^n(x_t-\bar{x})^2]$, where $x_t = \Sigma_{j=1}^t e_t$, $e_t \sim NID(0,1)$, $\bar{x}$ is the sample mean of the x_t's and $n = 1{,}000$. The results are presented in Table 2.

Table 2: $p = Prob(X \le x)$

p	x	p	x	p	x	p	x	p	x
0.01	-19.60	0.21	-7.63	0.41	-4.98	0.61	-3.26	0.81	-1.98
0.02	-17.08	0.22	-7.44	0.42	-4.87	0.62	-3.19	0.82	-1.92
0.03	-15.57	0.23	-7.26	0.43	-4.77	0.63	-3.12	0.83	-1.86
0.04	-14.33	0.24	-7.13	0.44	-4.67	0.64	-3.05	0.84	-1.81
0.05	-13.43	0.25	-6.98	0.45	-4.58	0.65	-2.99	0.85	-1.75
0.06	-12.66	0.26	-6.82	0.46	-4.47	0.66	-2.91	0.86	-1.67
0.07	-12.07	0.27	-6.68	0.47	-4.38	0.67	-2.85	0.87	-1.61
0.08	-11.53	0.28	-6.57	0.48	-4.30	0.68	-2.79	0.88	-1.55
0.09	-11.12	0.29	-6.41	0.49	-4.20	0.69	-2.73	0.89	-1.48
0.10	-10.73	0.30	-6.28	0.50	-4.10	0.70	-2.69	0.90	-1.41
0.11	-10.34	0.31	-6.16	0.51	-4.01	0.71	-2.62	0.91	-1.34
0.12	-9.96	0.32	-6.03	0.52	-3.93	0.72	-2.55	0.92	-1.28
0.13	-9.57	0.33	-5.90	0.53	-3.85	0.73	-2.48	0.93	-1.21
0.14	-9.29	0.34	-5.76	0.54	-3.78	0.74	-2.41	0.94	-1.14
0.15	-9.02	0.35	-5.64	0.55	-3.70	0.75	-2.35	0.95	-1.07
0.16	-8.76	0.36	-5.53	0.56	-3.62	0.76	-2.29	0.96	-0.99
0.17	-8.50	0.37	-5.42	0.57	-3.53	0.77	-2.21	0.97	-0.92
0.18	-8.28	0.38	-5.32	0.58	-3.47	0.78	-2.15	0.98	-0.81
0.19	-8.04	0.39	-5.21	0.59	-3.39	0.79	-2.08	0.99	-0.67
0.20	-7.83	0.40	-5.10	0.60	-3.33	0.80	-2.03		

3. DISCUSSION AND CONCLUSION

The results in Theorem 1 indicate that determining the presence of a unit root in a time series by looking at the sample autocorrelation function may be misleading. Also in the unit root case the sample autocorrelation function $r_n(m)$ may tail off rather quickly, due to the fact that for relatively large lag lengths m, $r_n(m)$ is approximately distributed as $1+(m/n)X$, where X is negative random variable with median -4.1 (cf. Table 2). Take for example the value of $r_n(m)$ for $m = 36$ in Table 1, i.e., $r_n(36) = 0.18$. It is not unreasonable to assume that $c(36) \approx 1$, so that by part (a) of Theorem 1, $r_n(m)$ is approximately distributed as $1 + (36/324)X$. From Table 2 it follows now that $P[r_n(36)<0.18] \approx P[X<-7.4] \approx 0.22$. Thus, the value $r_n(36) = 0.18$ is not unlikely under the unit root hypothesis.

Theorem 1 also suggests the following unit root test: Choose an m converging to infinity at order $o(n^{1/3})$, say $m = [cn^{\delta}]$ with $c > 0$ and $\delta \in (0,1/3)$ a priori chosen constants, and reject the unit root hypothesis at say 5% significance if $X_n(m) \leq -13.43$. Choosing $c = 10$, $\delta = 0.2$, this test yields for the US inflation rate: $m = 31$, $X_n(31) = -8.09$, hence again we cannot reject the unit root hypothesis at the 5% significance level. However, the asymptotic power of this test is inferior to the asymptotic power of the Phillips (1987) and Phillips-Perron (1988) tests Z_{α}. Consider a similar alternative hypothesis as in Phillips and Perron (1988), i.e., H_1: $y_t = \beta + u_t$, where (u_t) satisfies the conditions of Assumption 1. Then for $m \to \infty$ at rate $o(n^{1/3})$, $X_n(m)$ converges to $-\infty$ at a lower rate than Z_{α}, i.e., $\text{plim}_{n\to\infty} mX_n(m)/n = -1$ and $\text{plim}_{n\to\infty} Z_{\alpha}/n \in (-\infty,0)$. This argument corroborates our conclusion: don't use higher-order sample autocorrelations in determining the presence of a unit root.

APPENDIX: Proof of Lemma 1

Parts (a), (b) and (c) of Lemma 1 follow from Phillips (1987) and Phillips and Perron (1988). The other parts are proved in this Appendix.

First, let us derive some preliminary results. Let f be a continuous function on R and let $m/n \to 0$ as $n \to \infty$. For arbitrary $\varepsilon > 0$ and $\eta \in (0,1)$ we have

$$\text{(A.1)}\quad \limsup_{n\to\infty} P[|(1/m)\Sigma_{t=1}^{m} f(W_n(t/n)) - f(W_n(0))| > \varepsilon]$$

$$\le \limsup_{n\to\infty} P[\sup_{0\le\delta\le m/n}|f(W_n(\delta))-f(W_n(0))| > \varepsilon]$$

$$\le \lim_{n\to\infty} P[\sup_{0\le\delta\le\eta}|f(W_n(\delta))-f(W_n(0))| > \varepsilon]$$

$$= P[\sup_{0\le\delta\le\eta}|f(W(\delta))-f(W(0))| > \varepsilon] \to 0 \text{ as } \eta \to 0,$$

where the equality follows from (2.4). Hence

$$\text{(A.2)}\quad (1/m)\Sigma_{t=1}^{m} f(W_n(t/n)) = f(W_n(0)) + o_p(1) \Rightarrow f(0).$$

Similarly, we have:

$$\text{(A.3)}\quad (1/m)\Sigma_{t=n-m+1}^{n} f(W_n(t/n)) = f(W_n(1)) + o_p(1) \Rightarrow f(W(1)).$$

Proof of Part (d)

Let $S_t = \Sigma_{j=1}^{t} u_j$. Since $y_t = y_0 + S_t$ and $W_n(r) = S_{[rn]}/(\sigma\sqrt{n})$, we have

$$\text{(A.4)}\quad (1/(mn))\Sigma_{t=m+1}^{n}(y_t - y_{t-m})y_{t-m}$$

$$= (1/(mn))\Sigma_{t=m+1}^{n}(S_t - S_{t-m})S_{t-m} + (1/(mn))\Sigma_{t=m+1}^{n}(S_t - S_{t-m})y_0$$

$$= (1/(mn))\Sigma_{t=m+1}^{n}(S_t - S_{t-m})S_{t-m}$$

$$+ y_0(\sigma/\sqrt{n})(1/m)\Sigma_{t=n-m+1}^{n} W_n(t/n) - y_0(\sigma/\sqrt{n})(1/m)\Sigma_{t=1}^{m} W_n(t/n)$$

$$= (1/(mn))\Sigma_{t=m+1}^{n}(S_t - S_{t-m})S_{t-m} + O_p(1/\sqrt{n})$$

The latter conclusion follows from (A.2) and (A.3) with $f(x) = x$. Moreover, since $(S_t - S_{t-m})^2 = S_t^2 + S_{t-m}^2 - 2S_tS_{t-m}$, we have

$$\text{(A.5)}\quad (1/(mn))\Sigma_{t=m+1}^{n}(S_t - S_{t-m})S_{t-m}$$

$$= \tfrac{1}{2}(1/(mn))\Sigma_{t=m+1}^{n}[S_t^2 - S_{t-m}^2 - (S_t - S_{t-m})^2],$$

$$= \tfrac{1}{2}(1/m)\Sigma_{t=0}^{m-1}S_{n-t}^2/n - \tfrac{1}{2}(1/m)\Sigma_{t=1}^{m}S_t^2/n - \tfrac{1}{2}(1/(mn))\Sigma_{t=m+1}^{n}(S_t - S_{t-m})^2$$

$$= \tfrac{1}{2}\sigma^2(1/m)\Sigma_{t=n-m+1}^{n}W_n(t/n)^2 - \tfrac{1}{2}\sigma^2(1/m)\Sigma_{t=1}^{m}W_n(t/n)^2$$

$$- \tfrac{1}{2}(1/(mn))\Sigma_{t=m+1}^{n}(S_t - S_{t-m})^2$$

It follows from (A.2) and (A.3), with $f(x) = x^2$, that

$$\text{(A.6)}\quad (1/m)\Sigma_{t=n-m+1}^{n}W_n(t/n)^2 \Rightarrow W(1)^2,$$

$$\text{(A.7)}\quad (1/m)\Sigma_{t=1}^{m}W_n(t/n)^2 \Rightarrow W(0)^2 = 0.$$

We shall now prove that $(1/(mn))\Sigma_{t=m+1}^{n}(S_t - S_{t-m})^2$ converges in probability to $\sigma^2(m)$ if m is constant and to σ^2 if $m \to \infty$ at rate $o(^{1/3})$. Observe that

$$\text{(A.8)}\quad \Sigma_{t=m+1}^{n}(S_t - S_{t-m})^2 = \Sigma_{t=m+1}^{n}(\Sigma_{j=t-m+1}^{t}u_j)^2 = \Sigma_{i=1}^{m}\Sigma_{j=1}^{m}\Sigma_{t=1}^{n-m}u_{t+i}u_{t+j}.$$

Since u_t is α-mixing, $u_{t+i}u_{t+j}$ is α_{ij}-mixing, with

$$\text{(A.9)}\quad \alpha_{ij}(\ell) = 1 \text{ if } \ell < |i-j|,\ \alpha_{ij}(\ell) = \alpha(\ell - |i-j|) \text{ if } \ell \ge |i-j|.$$

Moreover, $\sup_t E|u_t|^\beta < \infty$ implies $\sup_{t,i,j}E|u_{t+i}u_{t+j}|^{\frac{1}{2}\beta} < \infty$. Similarly to the argument in Phillips (1987) it therefore follows that there exists a constant C not depending on t, i and j such that

$$\text{(A.10)} \quad |\mathrm{cov}(u_{t+k+i}u_{t+k+j}, u_{t+i}u_{t+j})| \le C\alpha(k-|i-j|)^{1-4/\beta} \text{ if } k \ge |i-j|,$$

$$\le C \text{ if } k < |i-j|,$$

hence for $1 \le i, j \le m$ and some constant C_*,

$$\text{(A.11)} \quad \Sigma_{k=0}^{\infty} |\mathrm{cov}(u_{t+k+i}u_{t+k+j}, u_{t+i}u_{t+j})| \le C|i-j| + C\Sigma_{k=0}^{\infty}\alpha(k)^{1-4/\beta} \le C_* m.$$

From this results it easily follows

$$\text{(A.12)} \quad E|(1/n)\Sigma_{t=1}^{n-m}(u_{t+i}u_{t+j} - Eu_{t+i}u_{t+j})|$$

$$\le \{E|(1/n)\Sigma_{t=1}^{n-m}(u_{t+i}u_{t+j} - Eu_{t+i}u_{t+j}|^2\}^{1/2} \le (C_* m/n)^{1/2} = O[\sqrt{(m/n)}],$$

hence

$$\text{(A.13)} \quad (1/(mn))\Sigma_{t=m+1}^{n}(S_t - S_{t-m})^2 = \sigma^2(m) + O_p[(m\sqrt{m}/\sqrt{n})].$$

Part (d) of Lemma 1 follows now from (2.6), (A.4) through (A.7) and (A.13).

Proof of Part (e)

Part (e) follows from (A.3), (A.4) and the easy equality

$$\text{(A.14)} \quad (1/(m\sqrt{n}))\Sigma_{t=m+1}^{n}(y_t - y_{t-m}) = (1/(m\sqrt{n}))\Sigma_{t=m+1}^{n}(S_t - S_{t-m})$$

$$= \sigma(1/m)\Sigma_{t=n-m+1}^{n}W_n(t/n) - \sigma(1/m)\Sigma_{t=1}^{m}W_n(t/n)$$

Proof of Part (f)

Observe that

(A.15) $(n/m)(d_n^2(m)-1) =$

$$= (n/n)[-(1/m)\Sigma_{t=n-m+1}^{n} S_t^2/n + 2(\bar{y}/\sqrt{n})(1/m)\Sigma_{t=n-m+1}^{n} S_t/\sqrt{n}$$

$$+ (1/m)\Sigma_{t=1}^{m} S_t^2/n - 2(\bar{y}/\sqrt{n})(1/m)\Sigma_{t=1}^{m} S_t/\sqrt{n}] \;/\; [(1/n)^2\Sigma_{t=m+1}^{n}(y_t-\bar{y})^2]$$

$$= [-\sigma^2(1/m)\Sigma_{t=n-m+1}^{n} W_n(t/n)^2 + 2\sigma(\bar{y}/\sqrt{n})(1/m)\Sigma_{t=n-m+1}^{n} W_n(t/n)$$

$$+ \sigma^2(1/m)\Sigma_{t=1}^{m} W_n(t/n) - 2\sigma(\bar{y}/\sqrt{n})(1/m)\Sigma_{t=1}^{m} W_n(t/n)]$$

$$/ \; [(1/n)^2\Sigma_{t=m+1}^{n}(y_t-\bar{y})^2].$$

It follows from (A.2), (A.3), (A.15), parts (a) and (c) of Lemma 1 and the fact that $W(0) = 0$,

(A.16) $$(n/m)(d_n^2(m)-1) \Rightarrow \frac{-W(1)^2 + 2W(1)\int_0^1 W(r)dr + W(0)^2 - 2W(0)\int_0^1 W(r)dr}{\int_0^1 W(r)^2 dr - [\int_0^1 W(r)dr]^2}$$

$$= \frac{-W(1)^2 + 2W(1)\int_0^1 W(r)dr}{\int_0^1 W(r)^2 dr - [\int_0^1 W(r)dr]^2}.$$

Part (f) of Lemma 1 follows now easily from (A.16).

Proof of Part (g)

It is easy to verify from Phillips (1987) and Phillips-Perron (1988) that

(A.17) $\bar{y}/\sqrt{n} = \sigma\int_0^1 W_n(r)dr + o_p(1);$

(A.18) $(1/n^2)\Sigma_{t=m+1}^{n}(y_{t-m}-\bar{y})^2 = \sigma^2\{\int_0^1 W_n(r)^2 dr - [\int_0^1 W_n(r)dr]^2\} + o_p(1);$

(A.19) $(1/n^2)\Sigma_{t=m+1}^{n}(y_t-\bar{y})^2 = \sigma^2\{\int_0^1 W_n(r)^2dr-[\int_0^1 W_n(r)dr]^2\} + o_p(1);$

Moreover, it follows from (A.2), (A.3) and the proofs of parts (d), (e) and (f) that

(A.20) $(1/(mn))\Sigma_{t=m+1}^{n}(y_t-y_{t-m})y_{t-m} = \frac{1}{2}\sigma^2[W_n(1)^2-\sigma^2(m)/\sigma^2] + o_p(1),$

(A.21) $(1/(m/n))\Sigma_{t=m+1}^{n}(y_t-y_{t-m}) = \sigma W_n(1) + o_p(1),$

(A.22) $$(n/m)(d_n(m)-1) = \frac{-\frac{1}{2}W_n(1)^2+W_n(1)\int_0^1 W_n(r)dr}{\int_0^1 W_n(r)^2dr-[\int_0^1 W_n(r)dr]^2} + o_p(1).$$

Thus, stacking the left-hand side random variables in (A.17) through (A.22) into a vector Z_n, we can write $Z_n = \Phi(W_n) + o_p(1)$, where Φ is a continuous mapping from $(D[0,1],\rho)$ onto R^6. Applying (2.4), part (g) of Lemma 1 follows.

REFERENCES:

Billingsley,P. (1968): *Convergence of Probability Measures*. New York: John Wiley

Dickey,D.A. and W.A.Fuller (1979): "Distribution of the Estimators for Autoregressive Times Series with a Unit Root", *Journal of the American Statistical Association*,74, 427-431.

Dickey,D.A. and W.A.Fuller (1981): "Likelihood Ratio Statistics for Autoregressive Time Series with a Unit Root", *Econometrica*, 49, 1057-1072.

Dickey,D.A., D.P.Hasza and W.A.Fuller (1984): "Testing for Unit Roots in Seasonal Time Series", *Journal of the American Statistical Association*, 79, 355-367.

Evans,G.B.A. and N.E.Savin (1981): "Testing for Unit Roots: 1", *Econometrica*, 49, 753-779.

Evans,G.B.A. and N.E.Savin (1984): "Testing for Unit Roots: 2", *Econometrica*, 52, 1241-1269.

Fuller,W.A. (1976): *Introduction to Statistical Time Series*. New York: John Wiley.

Haldrup,N. and S.Hylleberg (1989): "Unit Roots and Deterministic Trends, with Yet Another Comment on the Existence and Interpretation of a

Unit Root in U.S. GNP.", Mimeo, Institute of Economics, University of Aarhus.

Hylleberg,S. and G.E.Mizon (1989): "A Note on the Distribution of the Least Squares Estimator of a Random Walk with Drift", *Economics Letters*, 29, 225-230.

Newey,W.K. and K.D.West (1987): "A Simple Positive Definite Heteroskedasticity and Autocorrelation Consistent Covariance Matrix", *Econometrica*, 55, 703-708.

Pantula,S.G. (1989): "Testing for Unit Roots in Time Series Data", *Econometric Theory*, 5, 256-271.

Phillips, P.C.B. (1987): "Time Series Regression with a Unit Root", *Econometrica*, 55, 277-301.

Phillips,P.C.B. and P.Perron (1988): "Testing for a Unit Root in Time Series Regression",*Biometrica*, 75, 335-346.

Said,S.E and D.A.Dickey (1984): "Testing for Unit Roots in Autoregressive-Moving Average of Unknown Order", *Biometrica*, 71, 599-607.

Schwert,G.W. (1989): "Tests for Unit Roots: A Monte Carlo Investigation" *Journal of Business and Economic Statistics*, 7, 147-159.

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