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Nijkamp, P.; Reggiani, A.

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SPATIAL INTERACTION AND INPUT-OUTPUT MODELS:
A DYNAMIC STOCHASTIC MULTI-OBJECTIVE FRAMEWORK

P. Nijkamp
A. Reggiani

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Spatial Interaction and Input-Output Models:
A Dynamic Stochastic Multi-objective Framework

Peter Nijkamp
Dept. of Economics
Free University
P.O. Box 7161
1007 MC Amsterdam

Aura Reggiani
Dept. of Mathematics
University of Bergamo
Via Salvecchio, 19
24100 Bergamo

PN/172/mvk



Abstract

This paper aims at providing new insights into dynamic spatial interaction and I-O analysis by linking a multiple objective programming approach to stochastic dynamic spatial allocation models. In this context, a multiple objective optimal control model is designed, which appears to lead to a generalized solution for conventional spatial interaction problems. Next, by using a Wiener (or Brownian) motion process model for the dynamic evolution of flows of origin, a new type of dynamic spatial interaction model is generated, for which the solution (based on Hamilton-Jacobs-Bellman equations and Lagrangean equations) appears to incorporate a generalized stochastic gravity model.



1. Introduction

Input-output (I-O) analysis has already a long history in the field of regional economics. Thanks to the path-finding work of Leontief and Isard in the fifties and sixties, I-O analysis has become a major analytical tool for spatial interactions and allocations in the field of commodity flows, not only at an (inter)national but also at an (inter)regional scale. Extensions toward environmental and energy analysis in the seventies are also noteworthy in the framework of spatial-sectoral allocation analysis.

The estimation of the entries of a matrix in allocation analysis, given (changes in) row and column totals, has been given much attention in the past decades. The conventional transportation model in linear programming, the Koopmans-Beckmann quadratic assignment model, the RAS-updating method in input-output analysis, the entropy-based spatial interaction model and the minimum information principle are all examples of assessment methods of matrix cells, when direct information on the values of these cells is lacking. These methods have often played an important role in input-output analysis (see for a survey among others Batten and Boyce, 1986).

Various attempts have been made to provide a synthesis of such methods based on different methodologies. For example, it has been shown by Coelho and Wilson (1977), Erlander (1977) and Evans (1973) that a linear cost-minimizing model in spatial interaction analysis may be regarded as a special limit case of the entropy approach. In general, however, it is clear that the use of different assessment principles implies essentially the use of different (i.e. conflicting) objective functions. This case of multiple objective models in spatial interaction analysis has recently been introduced with in an interesting article by Hallefjord and Jörnsten (1986).

The latter situation is also extremely important in an interregional I-O framework, as sometimes cost-minimizing principles (e.g. the Leontief-Strout model) are applied, while in other cases information-theoretic or gravity-based principles are used. Thus the potential offered by multiple objective models deserves a closer analysis.

Despite the rigour and consistency achieved in input-output analysis, usually a basic flaw remains, viz. its essentially static character. Fully dynamic I-O models are still very rare. This is especially surprising, because in a related field, viz. spatial interaction analysis, various attempts have been made at designing dynamic models (see for a review Nijkamp and Reggiani, 1986a). It is a well-known fact (see Batten, 1983) that (inter)regional I-O analysis is a particular type of spatial interaction analysis: both approaches deal

with the assessment of the entries of a matrix, given its row and column totals. Therefore, it may be interesting to use recent achievements in the area of spatial interaction analysis as a 'model' for exploring the possibilities of designing fully dynamic I-O models, as it is clear that dynamic spatial interaction models may be regarded as generalized I-O models. Especially if the recorded marginal transactions are subject to a dynamic change pattern, results from dynamic spatial interaction theory may be helpful. In particular the use of optimal control principles may be relevant in this context. This would also provide a framework for endogenizing the technical structure which is usually imposed as a fairly rigid framework in I-O analysis. In this sense, technological change or changes at the demand side, for instance, may also be dealt with, so that indirect implications for the spatial allocation of commodities can be more properly analyzed.

A third aspect of allocation and I-O analysis, which has up till now received far less attention, is the potential stochasticity of these models. Various elements in dynamic models cannot plausibly be regarded to be deterministic in nature, so that it is worth exploring the field of stochastic allocation models. This problem has in the past received some attention in input-output analysis (see Gerking, 1976), in allocation problems based on the master equation analysis (see De Palma and Lefèvre, 1984, Haag and Weidlich, 1986, Kanaroglou et al., 1986, and Weidlich and Haag, 1983) and in random utility theory (De Palma and Lefèvre, 1983, and Leonardi, 1985), but in the broader context of spatial interaction analysis this issue has often been neglected.

In the light of the previous remarks, this paper will center around 3 issues in (spatial) allocation analysis, viz. multiple objectives, dynamics and stochasticity. An attempt will be made at providing a coherent framework for analysing these 3 issues simultaneously. Therefore, the paper is organized as follows. In section 2 the problem of multiple objectives in (spatial) allocation analysis is discussed, followed by a presentation of a simple dynamic model based on optimal control analysis in section 3. Section 4 is devoted to stochasticity, especially in the framework of optimal control models. A synthesis containing a new formal approach to multiple objective, dynamic and stochastic models for allocation analysis is contained in section 5, while the paper is concluded with a research outlook.

2. Multiple Objective Spatial Interaction Analysis

In the present section we will start off with some standard results from conventional static spatial interaction analysis (see for a review Nijkamp and Reggiani, 1986a). The usual spatial interaction model based on the minimum information principle (or maximum entropy

principle) can be reformulated in the context of multiple objective programming analysis as follows:

$$\begin{array}{l}
 \max \omega = -\frac{1}{\beta} \sum_i \sum_j T_{ij} (\ln T_{ij} - 1) - \sum_i \sum_j c_{ij} T_{ij} \\
 \text{s.t.} \\
 \sum_j T_{ij} = O_i \quad \forall i \\
 \sum_i T_{ij} = D_j \quad \forall j
 \end{array} \quad (2.1)$$

where T_{ij} is the flow from i to j , c_{ij} its corresponding unit cost, O_i the origin total and D_j the destination total. These flows may represent commodity flows in I-O analysis, transport flows in transportation analysis, migration flows in demo-economic analysis, etc. The parameter β acts as a distance friction coefficient, as it can easily be demonstrated (see Wilson et al., 1981) that the solution of (2.1) is equal to:

$$T_{ij} = A_i B_j O_i D_j \exp(-\beta c_{ij})$$

with A_i and B_j balancing factors defined as:

$$A_i = \exp(-\beta \lambda_i) / O_i \quad (2.3)$$

and

$$B_j = \exp(-\beta \mu_j) / D_j \quad (2.4)$$

where λ_i and μ_j are the Lagrange multipliers associated with O_i and D_j respectively.

Model (2.1) may also be presented as a multiple objective model with 2 objective functions, viz.:

$$\begin{array}{l}
 \max \omega = -\alpha_1 \sum_i \sum_j T_{ij} (\ln T_{ij} - 1) - \alpha_2 \sum_i \sum_j c_{ij} T_{ij} \\
 \text{s.t.} \\
 \sum_j T_{ij} = O_i \\
 \sum_i T_{ij} = D_j
 \end{array} \quad (2.5)$$

where the weight coefficients α_1 and α_2 satisfy the usual addition conditions:

$$\alpha_1 + \alpha_2 = 1 \quad (2.6)$$

This is obviously a model with two conflicting objectives, viz. maximization of interactivity (measured by the entropy) and minimization of total interaction costs (see also Hallefjord and Jörnsten, 1986). Now, it is easily seen that - instead of (2.3) and (2.4) - we have the following solution for the balancing factors:

$$\left. \begin{aligned} A_i &= \exp(-\lambda_i / \alpha_1) / O_i \\ B_j &= \exp(-\mu_j / \alpha_1) / D_j \\ \beta &= \alpha_2 / \alpha_1 \end{aligned} \right\} \quad (2.7)$$

It is clear that the abovementioned model can easily be extended with alternative objective functions, although in that case it is generally impossible to derive elegant analytical solutions. However, by means of interactive numerical procedures a final solution is in principle still possible (see for further details also Rietveld, 1980).

What is the meaning of the above mentioned multi-objective model in the context of I-O analysis? The use of this model in an I-O framework would imply a compromise between the conventional Leontief-Strout gravity trade model and a cost minimizing pooling model for I-O flows. The first component of (2.1) is in agreement with a gravity (or entropy) specification and hence consistent with an information-theoretic methodology. The second component emerges from a linear programming approach assuming cost minimization. Consequently, the use of this multi-objective approach in estimating (changes in) I-O coefficients is more flexible and general than conventional uni-dimensional estimation criteria.

3. Dynamic Multiple Objective Spatial Interaction Analysis

The previous multiple objective programming analysis can be extended by including a dynamic systems equation describing the evolution of the spatial system concerned over time. It is evident that various ways for specifying a dynamic spatial interaction model geared to (2.1) can be chosen. In general, however, it is then extremely cumbersome to derive manageable analytical results. Let us consider here the following dynamic model:

$$\dot{O}_i = \gamma_i O_i + \delta_i \left(\sum_j T_{ji} - \sum_j T_{ij} \right) \quad (3.1)$$

where γ_i is the natural growth rate of place i , and δ_i is a parameter. It is assumed here that O_i is a state variable whose dynamic evolution is linearly dependent on the net push-out and pull-in effects of origin i . It should be noted that model (3.1) is among

others a member of the family of dynamic migration models proposed by Okabe (1979) and Sikdar and Karmeshu (1982), which can be specified as follows:

$$\dot{P}_i = \gamma_i P_i + \sum_j T_{ji} - \sum_j T_{ij} \quad (3.2)$$

where $P_i = P_i(t)$ is the population size of place i . In fact, if we assume that the total push-out effects from origin i are linearly dependent (through the parameter δ_i) on the population size in i , we obtain:

$$O_i = \delta_i P_i \quad (3.3)$$

or

$$\dot{O}_i = \delta_i \dot{P}_i \quad (3.4)$$

In the context of the previous general dynamic model one may again raise the question of the relevance of such a model in the framework of I-O analysis. As mentioned before, a basic flaw of many I-O models is the static nature of both the technical relationships and the column and row totals. In our present model, the assumption is essentially made that the column (or row) totals are not only determined by external forces, but also by internal dynamics of the cell entries. In other words, the change in these totals is inter alia a function of the cells estimated in a previous period. This change is then due to attraction and repulsion effects (either between regions or between sectors). Clearly, a model of this nature leads to a highly dynamic I-O pattern, which may be further analyzed by means of dynamic programming or optimal control methods.

Next by substituting (3.3) and (3.4) into (3.2), we clearly obtain the dynamic equation (3.1). The general idea of (3.1) is that the rate of change of marginal totals in a given place is influenced by its own state value and by the net push-pull effects exerted on that specific place. Then we may specify the following optimal control model (with T_{ij} as control variables) with 2 conflicting objective functions:

$$\begin{aligned}
 \max \omega_1 &= \int_0^T - \sum_i \sum_j T_{ij} (\ln T_{ij} - 1) dt \\
 \max \omega_2 &= \int_0^T - \sum_i \sum_j c_{ij} T_{ij} dt \\
 \text{s.t.} & \\
 \sum_j T_{ij} &= O_i \\
 \sum_i T_{ij} &= D_j \\
 \sum_i O_i &= \sum_j D_j \\
 \dot{O}_i &= \gamma_i O_i + \delta_i (\sum_j T_{ji} - \sum_j T_{ij})
 \end{aligned} \tag{3.5}$$

Obviously, the constraints in (3.5) are assumed to hold each time period. It is also assumed that $\gamma_i < \delta_i$. It is a well-known fact that maximization of the first objective function will generate more dispersion among the cell values of an interaction matrix, whereas maximization of the second function will lead to an orientation toward a limited number of corner solutions (see Nijkamp, 1977).

Multiple objective programming theory indicates that the solution of (3.5) has to be located on the efficiency frontier of the two objective functions. The set of efficient solutions can (in principle) be generated by a linear parametrization of these objective functions (see Nijkamp, 1980), so that the following Hamiltonian H and Lagrangean function L can be obtained for a constrained optimal control model:

$$H = \varepsilon_1 \left\{ - \sum_i \sum_j T_{ij} (\ln T_{ij} - 1) \right\} + \varepsilon_2 \left\{ - \sum_i \sum_j c_{ij} T_{ij} \right\} + \sum_i \psi_i \dot{O}_i \tag{3.6}$$

where ψ_i represents the costate variable (associated with O_i), and:

$$L = H + \sum_i \lambda_i (O_i - \sum_j T_{ij}) + \sum_j \mu_j (D_j - \sum_i T_{ij}) + \rho (\sum_j D_j - \sum_i O_i) \tag{3.7}$$

The first-order conditions for a constrained maximum are:

$$\left. \begin{aligned}
 \frac{\partial L}{\partial T_{ij}} &= 0 \\
 \frac{\partial L}{\partial O_i} &= -\dot{\psi}_i \\
 \frac{\partial H}{\partial \psi_i} &= \dot{O}_i
 \end{aligned} \right\} \tag{3.8}$$

while the first-order optimality conditions related to the first set of conditions in (3.8) are:

$$\frac{\partial L}{\partial T_{ij}} = -\varepsilon_1 \ln T_{ij} - \varepsilon_2 c_{ij} - \lambda_i - \mu_j - \delta_i \Psi_i + \delta_j \Psi_j = 0, \quad (3.9)$$

so that we find:

$$T_{ij} = \exp \{ (-\lambda_i - \mu_j - \delta_i \Psi_i + \delta_j \Psi_j - \varepsilon_2 c_{ij}) / \varepsilon_1 \} \quad (3.10)$$

Now it is straightforward to derive that:

$$T_{ij} = A_i G_i^{-1} O_i B_j G_j D_j \exp(-\beta c_{ij}) \quad (3.11)$$

where:

$$\left. \begin{aligned} A_i &= \exp(-\lambda_i / \varepsilon_1) / O_i \\ B_j &= \exp(-\mu_j / \varepsilon_1) / D_j \\ G_i &= \exp(\delta_i \Psi_i / \varepsilon_1) \\ G_j &= \exp(\delta_j \Psi_j / \varepsilon_1) \\ \beta &= \varepsilon_2 / \varepsilon_1 \end{aligned} \right\} \quad (3.12)$$

By redefining

$$\begin{aligned} \bar{A}_i &= A_i G_i^{-1} \\ \bar{B}_j &= B_j G_j \end{aligned} \quad (3.13)$$

we can derive the following elegant standard solution:

$$T_{ij} = \bar{A}_i \bar{B}_j O_i D_j \exp(-\beta c_{ij}) \quad (3.14)$$

which is equivalent to the following logit form:

$$p_{ij} = \frac{T_{ij}}{O_i} = \frac{\bar{B}_j D_j \exp(-\beta c_{ij})}{\sum_j \bar{B}_j D_j \exp(-\beta c_{ij})} \quad (3.15)$$

with p_{ij} representing the probability of a move from i to j . Solution (3.14) of our optimal control problem is unique, as we are dealing with a concave integrand.

The parameter β functions also as a weight factor according to (3.12): if for instance $\beta \rightarrow 0$, then $\varepsilon_2 \rightarrow 0$ so that in case of a negligible distance friction the contribution of the corresponding cost function also vanishes, and vice versa. It should be noted that if we consider the optimal control problem (3.5) without the standard constraints on origin and destination (i.e., exclusively a model with the dynamic equation for O_i), we will obtain an unconstrained spatial interaction model of the following type:

$$T_{ij} = G_i^{-1} G_j \exp(-\beta c_{ij}) \quad (3.16)$$

where G_i and G_j may be interpreted as potential attractors of i and j , respectively.

The previous optimal control approach (3.5) can again be extended by introducing general dynamic objective functions, but then less tractable analytical results will emerge (see also Nijkamp and Reggiani, 1986b).

4. Stochastic Dynamic Multiple Objective Spatial Interaction Analysis

In the present section an attempt will be made at introducing stochasticity in the model described in section 3. First of all, following Arnold (1974), we define a "stochastic process" as a "differential equation for random functions", so that dynamics is implicitly embodied in the concept of stochasticity. In particular, we assume here that the originally deterministic model (3.1) has a random component of the so-called "white noise" type, representing the statistical uncertainty on the marginal totals of our dynamic spatial interaction model (caused e.g. by stochastic external growth processes).

It is well-known (see Arnold, 1974) that this "white noise" is a "very useful mathematical idealization for describing random influences that fluctuate rapidly and hence are virtually uncorrelated for different instants of time". In this context we assume, instead of our dynamic equation (3.1), a stochastic differential equation that obeys a continuous Wiener or Brownian motion process (related to the "white noise"), which can be represented as follows (see also Kamien and Schwartz, 1981, Kushner, 1971, and Malliaris and Brock, 1982):

$$\begin{aligned} dO_i &= \{Y_i O_i + \delta_i (\sum_j T_{ji} - \sum_j T_{ij})\} dt + O_i \sigma_i dz_i \\ &= g_i dt + O_i \sigma_i dz_i \end{aligned} \quad (4.1)$$

where dz_i is the incremental change in a stochastic process z that

satisfies a Wiener process. It is clear that (4.1) is a time-dependent process which should essentially be written as:

$$dO_i(t) = g_i(t)dt + O_i(t)\sigma_i(t)dz \quad (4.2)$$

The formal expression (4.2) is called Itô's stochastic differential equation (with initial condition $O_i(0) = O_i^*$), where σ_i is the diffusion component of the stochastic process. Clearly, in a deterministic context, $\sigma_i = 0$.

The latter specification indicates that the expected rate of change ('drift') is in fact represented by $g_i(t)$, but in this case there is also a stochastic disturbance term, which we have assumed to be proportional to the origin size O_i . It should be noted that - although Wiener processes modelling stochasticity have been used before in economics, physics and biology - the addition of a stochastic term to the differential equations describing the dynamics of a spatial model has been proposed so far only by Sikdar and Karmeshu (1982) for a non-linear gravity migration model and by Vorst (1985) for an urban retail model. It is a well-known result from the literature that, for a Wiener process z and for any partition t_0, t_1, t_2 of the time interval, the random variables $z(t_1)-z(t_0), z(t_2)-z(t_1), z(t_3)-z(t_2), \dots$ (i.e., the incremental changes) are independently and normally distributed with mean zero and variances $t_1-t_0, t_2-t_1, t_3-t_2, \dots$, respectively.

As far as this approach in the framework of I-O analysis is concerned, it is worth noting that the source and nature of stochasticity in I-O models is fraught with uncertainties. The specification of a conventional probability distribution for uncertain elements is then a less meaningful approach. By using a general Wiener process, we are able to encapture random changes in both the average pattern of column or row totals and in their dispersion.

Next, it can be shown (see Annex A) that without further constraints the solution of (4.1) is equal to:

$$O_i = F(t,z) = O_i^* \exp \{ (-\delta_i - \sigma_i^2/2)t + \sigma_i z_i + [\gamma_i + (\delta_i/O_i) \sum_j T_{ji}]t \} \quad (4.3)$$

so that (4.3) is the optimal trajectory of the state variable O_i .

Assuming again two objective functions (viz., an entropy function and a cost function), we may specify the following parametrized optimal control model, in which the mathematical expectation E of the weighted objective functions have to be maximized:

$$\begin{aligned}
\omega &= \max E \int_0^T -[\epsilon_1 \{ \sum_i \sum_j T_{ij} (\ln T_{ij} - 1) \} + \epsilon_2 \sum_i \sum_j c_{ij} T_{ij}] dt \\
\text{s.t.} & \\
\sum_j T_{ij} &= O_i \\
\sum_i T_{ij} &= D_j \\
\sum_i O_i &= \sum_j D_j \\
dO_i &= (\gamma_i O_i + \delta_i (\sum_j T_{ji} - \sum_j T_{ij})) dt + O_i \sigma_i dz_i = g_i dt + O_i \sigma_i dz_i
\end{aligned} \tag{4.4}$$

We will now analyse the latter model by first regarding the constraints on the control variables T_{ij} . Following Malliaris and Brock (1982), we may write the Hamilton-Jacobi-Bellman equation associated with (4.4) as:

$$\begin{aligned}
-\frac{\partial \omega}{\partial t} &= \max_{T_{ij}} (-[\epsilon_1 \{ \sum_i \sum_j T_{ij} (\ln T_{ij} - 1) \} + \epsilon_2 \sum_i \sum_j c_{ij} T_{ij}] + \\
&\quad \sum_i \frac{\partial \omega}{\partial O_i} g_i + 1/2 \sum_i \frac{\partial^2 \omega}{\partial O_i^2} (\sum_j T_{ij})^2 \sigma_i^2)
\end{aligned} \tag{4.5}$$

where the assumption is made that O_i is uncorrelated with O_j . Equation (4.5) can now also be written as:

$$-\frac{\partial \omega}{\partial t} = \max_{T_{ij}} H^* \tag{4.6}$$

where H^* is the functional form of the expression in brackets at the right hand side of (4.5).

Next, if we define the costate variable $\psi_i^*(t)$ as follows:

$$\psi_i^*(t) = \frac{\partial \omega}{\partial O_i} \tag{4.7}$$

we may write H^* as:

$$\begin{aligned}
H^* &= -[\epsilon_1 \{ \sum_i \sum_j T_{ij} (\ln T_{ij} - 1) \} + \epsilon_2 \sum_i \sum_j c_{ij} T_{ij}] + \\
&\quad \sum_i \psi_i^* g_i + 1/2 \sum_i \frac{\partial \psi_i^*}{\partial O_i} \sigma_i^2 (\sum_j T_{ij})^2
\end{aligned} \tag{4.8}$$

Then by comparing (4.8) with (3.6) we can derive:

$$H^* = H + 1/2 \sum_i \frac{\partial \psi_i^*}{\partial O_i} \sigma_i^2 (\sum_j T_{ij})^2 \quad (4.9)$$

where the last term at the right hand side of (4.9) represents thus the stochastic part of the dynamic process concerned.

Now we will introduce the constraints on the control variables. Then it is straightforward (see Chow, 1979) to define the following Lagrangean expression:

$$L^* = H^* + \sum_i \lambda_i (O_i - \sum_j T_{ij}) + \sum_j \mu_j (D_j - \sum_i T_{ij}) + \rho (\sum_j D_j - \sum_i O_i) \quad (4.10)$$

In the latter case we may apply the Pontryagin Stochastic Maximum Principle (see Malliaris and Brock, 1982). This principle states that for an optimal control variable T_{ij}^* that maximizes the Lagrangean (4.10) the following conditions hold:

- the costate function ψ_i^* satisfies the following stochastic differential equation:

$$d\psi_i^* = - \frac{\partial L^*}{\partial O_i} dt + \sum_i \frac{\partial \psi_i^*}{\partial O_i} O_i \sigma_i dz_i \quad (4.11)$$

- the following transversality condition holds

$$\left. \begin{aligned} \psi_i^* \{O_i(T), T\} &= \frac{\partial \omega}{\partial O_i} \{O_i(T), T\} \geq 0 \\ \psi_i^* (T) O_i(T) &= 0 \end{aligned} \right\} \quad (4.12)$$

Furthermore, it is easily seen that the optimal solution T_{ij}^* is equal to:

$$T_{ij}^* = \exp \{ (-\lambda_i - \mu_j - \delta_i \psi_i^* + \delta_j \psi_j^* - \epsilon_2 c_{ij} + O_i \sigma_i^2 \frac{\partial \psi_i^*}{\partial O_i}) / \epsilon_1 \} \quad (4.13)$$

This expression leads thus to the interesting result that - apart from the last stochastic term - the same formal outcome is obtained as in (3.10), so that the final solution is:

$$T_{ij}^* = A_i^* O_i Z_i B_j^* D_j \exp(-\beta c_{ij}) \quad (4.14)$$

where A_i^* and B_j^* have similar formal definitions as in (3.13), i.e.:

$$\begin{aligned} A_i^* &= A_i \exp(\delta_i \Psi_i^* / \epsilon_1)^{-1} \\ B_j^* &= B_j \exp(\delta_j \Psi_j^* / \epsilon_1) \end{aligned} \quad (4.15)$$

and where Z_i is equal to:

$$Z_i = \exp(O_i \sigma_i^2 \frac{\partial \Psi_i^*}{\partial O_i} / \epsilon_1) \quad (4.16)$$

It should be noted that here A_i^* and B_j^* are stochastic terms as they incorporate the costate variable Ψ_i^* satisfying (4.11) and (4.12). Obviously, Ψ_i^* does not have an explicit solution owing to the difficulties involved in the calculations of (4.5) and (4.11). It should also be added that the term Z_i is stochastic because it incorporates the diffusion component σ_i^2 of the stochastic Wiener process z_i .

It is also interesting to see that the stochastic solution (4.14) can formally be transformed into a logit form as follows:

$$P_{ij}^* = \frac{T_{ij}^*}{O_i} = \frac{1}{Z_i} \cdot \frac{B_j^* D_j \exp(-\beta c_{ij})}{\sum_j B_j^* D_j \exp(-\beta c_{ij})} \quad (4.17)$$

which is in agreement with standard results from conventional spatial interaction analysis. In fact if $\sigma_i = 0$, eq. (4.17) yields as a special case the logit form (3.15) obtained from the deterministic approach. Consequently, introduction of a stochastic white noise process in interaction and I-0 analysis is in principle possible and does not affect the basic structure of a gravity type of solution.

5. Concluding Remarks

The main result emerging from the previous analysis is that stochastic fluctuations tend to de-stabilize a spatial interaction system.

This result is fully in agreement with a recent stochastic analysis of a spatial interaction problem, proposed by Sikdar and Karmeshu (1982), where, however, the stochastic process is solved via a so-called Stratonovic prescription. In particular the introduction of random perturbations of the "white noise" type in the rate of change of the exits O_i sheds new light on stochastic process models analyzed so far (see for an interesting review also Pickles, 1980, and Kanaroglou et al., 1986). If we consider the dynamic probability P_{ij}^* as a transition probability from a state i to a state j , as given by eq. (4.17), it can be decomposed (see also Cordey-Hayes and Gleave, 1974, and Tomlin, 1979) into an "escape frequency" (determined

by the term $1/Z_i$) and a "capture probability" (determined by the stochastic logit form). In our context the "escape frequency" is a function depending on the variance σ_i^2 which, according to Cordey-Hayes and Gleave's hypothesis, might be caused by generic factors like age distribution of population, variation of income with time, etc. (or, in case of I-O analysis, by sectoral developments); the "capture probability" incorporates implicitly the stochasticity via the term B_j^* .

From a theoretical viewpoint the above concept might have some resemblance with fluctuations in biological models. Here, it is well-known (see Maynard Smith, 1974) that the effect of the environmental (external) fluctuations is to change the level prevailing in a deterministic framework (for example, the case of very strong fluctuations causing the extinction of a population).

Another important result emerging from our analysis is that the solution is formally compatible with a logit expression in both a deterministic and a stochastic approach. On the one hand this confirms also the descriptive and explanatory power of logit models (and hence implicitly also of macro approaches related to the maximization of entropy) in a dynamic context (see Nijkamp and Reggiani, 1986a). On the other hand this may offer especially appealing perspectives in the field of empirical applications. In this context it is interesting to recall the contribution by Haag (1986). This author introduces stochasticity via a master equation approach and ends up with the same structure of the stationary flow distribution as static random utility theory.

Finally, in the framework of multi-objective analysis equation (4.17) shows which are the efficient (stochastic) solutions, emerging from two conflicting objectives varying over time and subject to a stochastic evolution of the state variables. Although multi-objective analysis has been used in dynamics before (see for instance Nijkamp, 1977, 1979 and 1980), its use in a stochastic context is still limited so far (see Ermoliev and Leonardi, 1981); the analysis presented in section 4 tries to make a new theoretical effort in the field of spatial interaction and I-O analysis.

The merger of the above mentioned classes of I-O models and spatial interaction models appears to lead to new theoretical insights regarding the stability of spatial-economic systems. At the more practical level of empirical I-O analysis, it is worth mentioning that the use of a more general multi-objective method for estimating (changes in) cells consequent upon changes in column or row totals may lead to a higher flexibility and a plausible range of estimates (instead of single point estimates). This is also extremely important to test the long-run stability of results of input-output models, as was shown in our optimal control approach. And finally, the use of white noise

random processes allows the researcher to include stochastic processes in a more flexible way than is being done in conventional probabilistic approaches.

In conclusion, we may stress the importance of stochasticity in the evolution of a spatial interaction system by showing, through equation (4.17) how the fluctuations may influence the standard results, in both a theoretical and empirical respect. In this context, I-O models can be represented in a generalized structural form incorporating stochastic dynamics.

Annex A. Proof of the Solution of a Wiener Process
for a Dynamic Spatial Interaction Model

By starting off from (4.3) and using Itô's theorem, we have:

$$dF(z,t) = F_t dt + F_z dz + 1/2 F_{zz} (dz)^2 \quad (A.1)$$

and

$$\frac{dz}{dt} \left| \begin{array}{cc} dz & dt \\ dt & 0 \end{array} \right. \quad (A.2)$$

so that:

$$F_t = (-\delta_i - \sigma_i^2 / 2 + \gamma_i) O_i + \delta_i \sum_j T_{ji} \quad (A.3)$$

$$F_{z_i} = O_i \sigma_i \quad (A.4)$$

$$F_{z_i z_i} = O_i \sigma_i^2 \quad (A.5)$$

By substituting now (A.2), (A.3), (A.4) and (A.5) into (A.1), we can easily check that (4.1) is consistent with the original Wiener process:

$$\begin{aligned} dO_i &= \{-O_i \delta_i - (\sigma_i^2 / 2) O_i + (\sigma_i^2 / 2) O_i + \gamma_i O_i + \delta_i \sum_j T_{ji}\} dt + O_i \sigma_i dz_i \\ &= \{\gamma_i O_i + \delta_i (\sum_j T_{ji} - O_i)\} dt + O_i \sigma_i dz_i \quad (A.6) \\ &= \{\gamma_i O_i + \delta_i (\sum_j T_{ji} - \sum_j T_{ij})\} dt + O_i \sigma_i dz_i \end{aligned}$$

Q.E.D.

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