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Lessons from Non-Linear Dynamic Economics

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Abstract

In dynamic economic analyses, low order linear models have traditionally dominated theoretical and empirical work. In such analyses, observed major shocks must be treated as exogenous forces and outside the realm of prediction. Dissatisfaction with this approach has led in recent years to the use of non-linear dynamic systems as a means of endogenising irregular and volatile behaviour, commonly observed in many economic situations.

This paper focusses on the predictive ability of such non-linear dynamic systems. Specifically, the implications of the theory of chaos for economic modelling are addressed by means of two illustrative models of economic development in discrete time. The first model generates growth by exogenous accumulation of conventional production factors and social overhead capital, while the second allows for endogenous technological change. In both cases, the system is constrained by bottleneck phenomena. It is found that for plausible parameter values chaotic regimes are unlikely although increasing the unit of time in the former model, and thereby amplifying the magnitude of change in the inputs to production, increases the likelihood of chaotic development.

In empirical work, stochastic noise can often not be distinguished from non-linear determinism and future research will need to focus on identifying the explicit form of the non-linear, but not necessarily chaotic, deterministic structure which may be hidden in many types of economic data.

Keywords:

Non-linear dynamics, chaos, economic growth, bottlenecks.



1. Introduction

Our economic world is highly dynamic and exhibits a wide variety of fluctuating patterns. This forms a sharp contrast with our current economic toolbox, which is largely filled with linear and comparative static instruments. Clearly, linear economic models do not necessarily generate stable solutions, but their evolution is only capable of generating four types of time paths: oscillatory and stable; oscillatory and explosive; monotonic and stable; and monotonic and explosive. This is true for linear models of any order, so that such models are only able to generate a limited spectrum of dynamic behaviour. Non-periodic evolution for instance, can normally not be described by our analytical apparatus, unless stochastic processes describing non-linear transition processes are assumed (see Brock 1986, Priestley, 1988, and Schuster 1984).

Non-linear dynamic relationships in economics are certainly not an unknown phenomenon and Goodwin's business cycle model of the 1950s is a well known example (see also Goodwin, 1982), but in most empirical applications linear (or linearized) models are still dominant. One important reason is that **non-linear dynamic econometrics** is by no means a well developed field of research and another is that **specification theory** is still a weak part in economic modeling (see Blommestein, 1986). In general, the issue of non-linear dynamics in economic modeling is less interesting when it concerns stochastic properties of the system, but much more when it concerns the way synchronic and diachronic processes are intertwined (see also Barnett et al., 1990, Lichtenberg and Lieberman, 1983, Liossatos, 1980, and Turner, 1980). Discontinuities in a system's behaviour may then emerge under certain conditions, which reflects essentially a morphogenesis in the evolution of the system concerned. Such morphogenesis may be based on either endogenous forces (e.g., behavioural feedbacks, overlapping generations), or exogenous forces (e.g., in the case of random shock models or ceilings and floors models) or a combination of both (e.g., regime switching models).

In recent years, a wide variety of dynamic economic models for countries, sectors or regions has been developed. Surprisingly enough, only a limited number of these studies exhibited **structural** dynamics. A major analytical problem in this respect is the question whether structural changes are caused by **intra-systemic** (endogenous) developments or exogenous forces (external to the system). This problem bears some similarities to the well known scientific debate on the existence of

long waves in economics, where especially the Schumpeterian viewpoint regarding the **endogeneity** of phases in a Kondratieff long wave is being tested (see also Grandmont, 1985, Kleinknecht, 1986).

In any case, a meaningful model for analyzing and predicting structural dynamics of an economic system should be able to generate various trajectories for the evolution of the system, in which both endogenous and exogenous fluctuating patterns may play a role. Furthermore, such a model may lead to testable hypotheses in order to explore under which conditions a stable development may emerge. In recent years, this has led to the popularity of the **theory of chaos**, especially since this new research line is focusing attention on the driving forces and trajectories of dynamic evolution.

In this paper we address the situations in which non-linear dynamics may arise in economic phenomena and the predictive ability of models which exhibit non-linear motion. In the next section we review the key issues in the theory of chaos. Since a number of comprehensive surveys of the economic applications of this theory in the last decade have been published recently (see Kelsey, 1988; Baumol and Benhabib, 1989; Boldrin and Woodford, 1991; Scheinkman, 1990; Radzicki, 1990; Rosser, 1991), our survey can be brief. However, we will illustrate the key issues by means of two models of economic development in discrete time. Section 3 describes a simple non-linear growth model in which growth is generated by exogenous accumulation of conventional production factors and social overhead capital. In this model, bottlenecks resulting from congestion and other externalities, generate decreasing returns to the conventional production factors. For plausible parameter values, this process leads to monotonic convergence to a stationary state. However, under the assumption that productivity shocks are discrete and lumpy, cyclic or chaotic motion may emerge.

In the second model, described in section 4, endogenous technological change provides a positive feedback to economic growth, but bottleneck phenomena again limit growth. Stylized facts regarding economic development give here little guidance about certain parameter values but it will be shown that the model could exhibit a wide range of dynamic behaviour.

In the last section we reflect on the implications for further work in this area.

2. Issues in Chaos Theory

Chaos theory has attracted widespread interest in the social sciences. In addition to the economic surveys mentioned earlier, informative reviews of chaos theory and its relevance for the social sciences can be found in among others, Andersen (1988), Benhabib and Day (1981, 1982), Boldrin (1988), Crilly et al., 1990, Devaney (1986), Guckenheimer and Holmes (1983), Lasota and Mackey (1985), Lung (1988), Pohjola (1981), Prigogine and Stengers (1985), Rosser (1991), Stewart (1989), and Stutzer (1980).

It is noteworthy that the new logic which has emerged in the area of non-linear dynamics by the introduction of the theory of chaos has also an interesting psychological appeal; model builders need not necessarily be blamed any more for false predictions, as errors in predictions may be a result of the system's complexity, as can be demonstrated by examining more carefully the properties of the underlying non-linear dynamic model. A fact is that chaos theory is currently regarded as a major discovery with a high significance for both the natural and social sciences.

An important feature of chaos theory is that it is essentially concerned with **deterministic, non-linear dynamic** systems which are able to produce **complex motions** of such a nature that they are sometimes seemingly random. In particular, they incorporate the feature that small uncertainties may grow exponentially (although all time paths are bounded), leading to a broad spectrum of different trajectories in the long run, so that precise or plausible predictions are - under certain conditions - very unlikely.

In this context, a very important characteristic of non-linear models which can generate chaotic evolutions is that such models exhibit strong sensitivity to initial conditions. Points which are initially close will on average diverge exponentially over time, although their time path is bounded and they may be from time to time briefly very close to each other. Hence, even if we knew the underlying structure exactly, our evaluation of the current state of the system is subject to measurement error and, hence it is impossible to predict with confidence beyond the very short run. Similarly, if we knew the current state with perfect precision, but the underlying structure only approximately, the future evolution of the system would also be unpredictable. The equivalence of the two situations has been demonstrated by e.g. Crutchfield et al. (1982).

An example of the extreme sensitivity of chaotic models to parameter values is demonstrated in Figure 1. This figure shows an example of the standard May (1976) model $X_{t+1} = a(1-X_t)X_t$ with the parameter $a=3.8$ and the initial value X_0 equal to the equilibrium point $1-\frac{1}{a}$ with as much precision as a modern high-speed computer allows. Figure 1 shows that under these circumstances the model exhibits stability for up to 50 periods, but the finite precision arithmetic of the computer generates after this point slight movements in X_t which are quickly amplified to chaotic fluctuations.

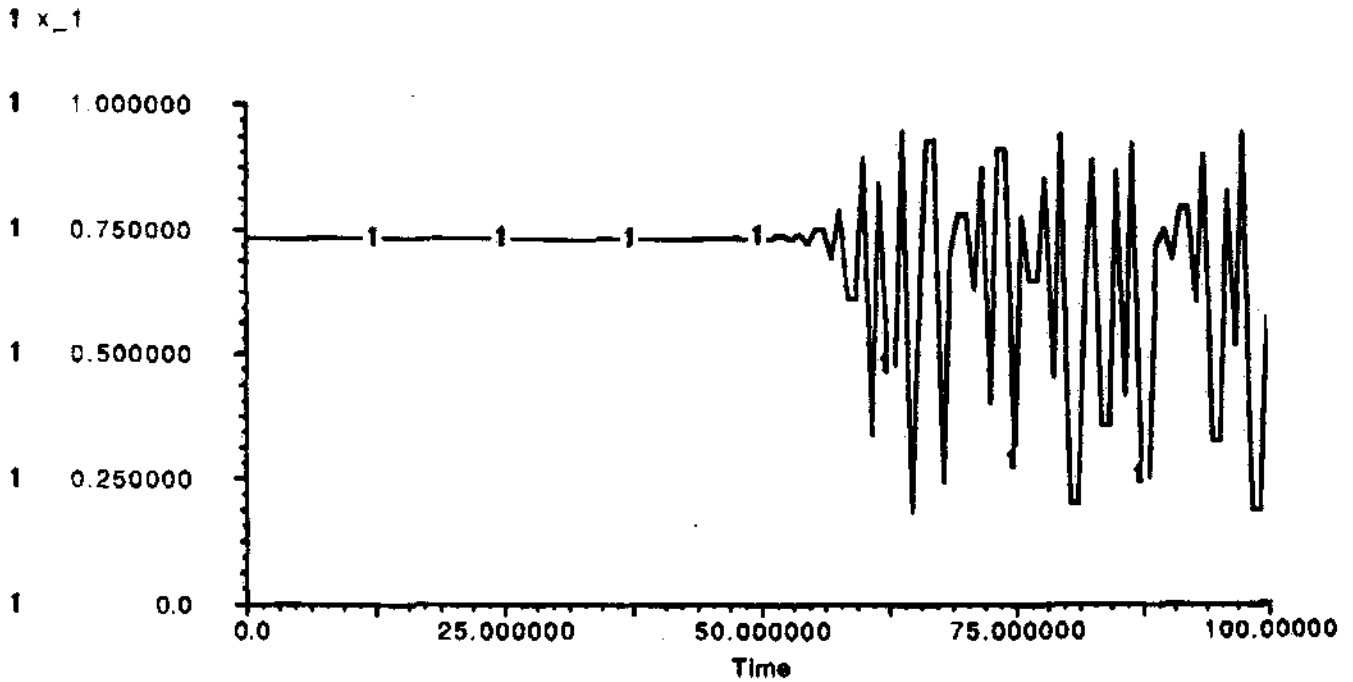


Figure 1. The standard May model.

After a series of interesting studies on chaotic features of complex systems in physics, chemistry, biology, meteorology and ecology, chaos theory has also been introduced and investigated in the field of economics and geography. The main purpose of the use of this theory in the social sciences was to obtain better insight into the underlying causes of unforeseeable evolutions of complex dynamic systems.

In recent years, the economics discipline has witnessed an increasing wave of contributions in the use of chaos theory for analyzing economic dynamics. We noted already in the introduction that in the

1950's an interesting application of chaos theory to economics (in particular, the existence of stable limit cycles in non-linear dynamics) was developed by Goodwin. He studied economic dynamics by means of an **accelerator-multiplier** framework for persistent, deterministic oscillations as an **endogenous** result of a dynamic economic system (see for a survey Goodwin, 1982). But only recently the awareness has grown that deterministic (periodic or a-periodic) fluctuations (or even bifurcations and jumps) in a complex dynamic economic system may be the result of small perturbations. Unexpected behaviour of non-linear dynamic models leads to the question of **validity of model specifications** (i.e., are model specifications compatible with plausible economic hypotheses) and of **testability of model results** (i.e., are model results - qualitatively or quantitatively - justifiable from possible non-linear patterns in the underlying data set). Further expositions on these issues can be found, amongst others, in Scheinkman (1990), Baumol and Benhabib (1989), Baumol and Quandt (1985), Chen (1988), Kelsey (1988) and Lorenz (1989). Applications and illustrations of chaos theory in economics can be found inter alia in the following fields:

- growth and business cycle theory (Balducci et al. 1984; Benhabib and Day 1982; Boldrin 1988; Day 1982; Funke 1987; Guckenheimer et al. 1977; Grandmont 1985, 1986; Hommes et al., 1990; Stutzer 1980).
- cobweb models (Chiarella 1988)
- long waves analysis (Nijkamp 1987; Rasmussen et al. 1985; Sterman 1985)
- R&D analysis (Baumol and Wolff 1983; Nijkamp et al. 1991)
- consumer behaviour (Benhabib and Day 1981)
- duopoly theory (Rand 1978; Dana and Montrucchio 1986)
- economic competition (Deneckere and Pelikan 1986; Ricci 1985)
- international trade (Lorenz 1987)
- competitive interactions between individual firms (Albin 1987)
- equilibrium theory (Hommes and Nusse, 1989; Nusse and Hommes, 1990).
- stock returns and exchange rates (LeBaron, 1989).

Interesting applications of chaos theory to related branches of economics can be found in geography and regional science (see for a survey Nijkamp and Reggiani, 1989, 1992). Examples here are:

- regional industrial evolution (White 1985)
- urban macro dynamics (Dendrinos 1984)
- spatial employment growth (Dendrinos 1986)

- relative population dynamics (Dendrinos and Sonis 1987)
- spatial competition and innovation diffusion (Sonis 1986, 1987)
- migration systems (Haag and Weidlich 1983, Reiner et al. 1986)
- urban evolution (Batty 1991, Nijkamp and Reggiani 1988)
- transport systems (Reggiani 1990)

It is interesting to observe that most applications of chaos theory in economics (and in general the social sciences) lack empirical content. While empirical research on chaos went hand in hand with theoretical developments in the natural sciences in the early 1980s, attempts to detect chaos in financial and economic data are more recent. The results in this area are so far disappointing. Brock (1989) claims that as yet no class of structural economic models has been **estimated** which allows for chaotic behaviour and in which the estimated model parameters are indeed in the chaotic range. Moreover, statistical tests which have been designed to detect chaos in time series without a **priori** specification of the nature of the data generating process, have not provided as yet unambiguous empirical support for the presence of chaos in observable economic processes.

The central concept for the statistical detection of chaos is that of **dimension** of the time series which can be loosely interpreted as the minimum number of lags that one would need to describe the dynamical behaviour of a time series in the long-run (see also Brock 1989). A very long truly random time series has a near-infinite dimension, but sequences of observations in a chaotic time series clump together in a lower dimensional space. Several statistical tests have been developed to test for the presence of such low-dimensional deterministic chaos, notably the Brock-Dechert-Scheinkman (1987) statistic, which has a standard normal distribution under the null hypothesis of pure randomness. This statistic detects a wide range of deviations from white noise in fluctuations rather than just chaos and is therefore a useful tool in specification analysis for estimation of time series models. Additional tools such as the largest Lyapunov exponent and recurrence plots are available (for details see Brock, 1988). However, so far the weight of evidence in range of economic and financial data is against the hypothesis that there is low-dimensional deterministic chaos in such time series. However, this does not imply that non-linear dynamic structure is absent from economic and financial time series but the available tests are not able to identify the nature of this structure. For example, Brock and Sayers (1988) found evidence of nonlinearity in the

following US statistics: employment and unemployment (quarterly), industrial production (monthly) and pigiron production (annually). Empirical evidence of nonlinear dynamics is now also emerging elsewhere, e.g. in weekly price observations in German agricultural markets (Finkenstädt and Kuhbier, 1990) and Austrian demographic data (Prskawetz, 1990). However, non-linear determinism tends to be absent in many macroeconomic aggregates such as GNP and private investment. One rare finding of chaos in monetary aggregates was recorded by Barnett and Chen (1988), but their finding has been convincingly challenged by Ramsey et al (1990).

There are at least four reasons why linear modelling of economic time series may be adequate and why a non-linear deterministic structure may therefore be absent or undetectable in such cases. First, farsighted economic agents have a desire to smooth consumption and production over time. This creates **negative feedback** loops. In contrast, it can be easily demonstrated that deterministic chaos requires **positive feedback** loops which may be found in phenomena such as industrial clustering, networking and the growth of cities, but which could be less likely in financial and economic time series (Brock, 1989). For example, the efficient markets hypothesis suggests that if deterministic structure could be detected in the **innovations** in e.g. share market data, such structure would vanish as agents would attempt to profitably exploit it in their forecasting of price movements (e.g. Fama 1976). Some evidence of low-dimensional chaos has nonetheless been found in time series on US stock returns (Scheinkman and LeBaron, 1989), although an analysis of e.g. New Zealand share market data suggested the opposite (Allen, 1989). A third reason for the difficulty in detecting chaos is that economic time series are, after detrending, inherently noisy due to measurement errors and outside shocks. In this case there may be some deterministic structure underlying the stationary fluctuations, but the high-dimensional chaos generated by this process may be indistinguishable from true randomness. Finally, the disappointing results to date may be explained by the focus on relatively short traditional macroeconomic series rather than long time series of microlevel data in which there may be more potential for nonlinear determinism. The power of the available tests may only be satisfactory when the available number of observations is of the order of 10^4 to 10^6 rather than 10^2 as is common practice, although it has to be added that recent statistical methods of dimension calculus and nonlinearity testing can get by with much smaller data sets (using

mainly Monte Carlo tests; see Hsieh, 1989 and Ramsey et al., 1990). In general however, suitable microlevel data sets are yet hardly available.

In the next two sections we will evaluate the use and relevance of chaos theory by means of two related examples, one in the field of economic restructuring (Section 3) and another one concerning the impact of innovation and R&D on diseconomies of scale (Section 4). In both cases it will be shown that in case of reasonable growth rates stable behaviour is likely to emerge, but that in case of (very) high growth suddenly unexpected fluctuations may emerge.

3. An Analysis of Evolutionary Economic Development

Following the conventional Hirschman (1958) paradigm we assume here that a proper combination of conventional **productive resources** and **public overhead capital** (including R&D) is a necessary condition for balanced growth. These factors are essentially the propulsive motives and incubators for the process of structural economic developments (see also Rosenberg 1976). It is plausible that in case of qualitative changes in a non-linear dynamic production system several shocks and perturbations may emerge (see for interesting illustrations also Allen and Sanglier 1979; Casetti 1981; Dendrinis 1981; and Wilson 1981). A simple mathematical representation of the driving forces of such a production system can be found in Nijkamp (1983, 1984, 1989). This simplified model was based on a so-called **quasi-production function** (including productive capital, infrastructure and R&D capital as arguments). The dynamics of the system were described by motion equations for productive investments, infrastructure investments and R&D investments. Several constraints (i.e., ceilings) were also added, for instance, due to the existence of capacity limits.

In our illustration we will start with a simple dynamic neo-classical production function as the basis for a more formal analysis of growth patterns of an economy. The assumption is made that output is generated by a mix of conventional production factors (capital, labour) and public overhead capital (including R&D capital). Later on we will turn to a more complicated and comprehensive economic system (Section 4) and also analyse the stability properties of that system. Here, the following Cobb-Douglas production function will be assumed for our (closed) production system:

$$Y = \alpha Q^\beta P^\gamma, \quad (3.1)$$

with Y , Q and P representing output, conventional production factors and social overhead capital, respectively. The parameters β and γ reflect the production elasticities concerned. It is well known that, if instead of social overhead and R&D capital an exponential growth rate of technological progress would have been included in (3.1), the resulting Cobb-Douglas production function would have been at the same time Harrod-, Hicks- and Solow-neutral, provided the technical change concerned would have been disembodied (see also Rouwendal and Nijkamp 1989, and Stoneman 1983).

A production function of type (3.1) may only be a reasonable approximation of the underlying production technology within a range of realistic **floors** and **ceilings** (Y_{\min} , Y_{\max}). Only in this range the production elasticities are assumed to be strictly positive. It is known from the literature on biological population dynamics (e.g. Pimm, 1982) that the existence of either floors or ceilings may generate fluctuating patterns. This is likely to be relevant also in an economic context. Below the minimum threshold level Y_{\min} , the critical mass of the economy may be too small to generate economies of scale and scope, so that then a marginal increase in one of the production factors may have a negligible impact on the net output of production. This situation suggests that an economy needs a minimum endowment with production factors before it reaches a self-sustained growth trajectory (see also McKenzie and Zamagni, 1991).

Furthermore, beyond a certain maximum capacity level Y_{\max} of the economy, bottleneck phenomena (congestion, diseconomies of scale of scope, e.g.) - caused by a high geographic or industrial concentration of Q - may again lead to a zero or even negative marginal product of conventional production factors. Any further increase in these production factors may then diminish output, unless this situation of a negative marginal product is compensated and corrected by the implementation of new public overhead and R&D investments (the well-known 'depression trigger' phenomenon in Schumpeterian theory).

It is easily seen that, if model (3.1) is explicitly put in a dynamic form, within the relevant range (Y_{\min} , Y_{\max}) the changes in output in a certain period of time may be approximated by means of the following discrete time version of (3.1):

$$\Delta Y_t = (\beta_t q_t + \gamma_t p_t) Y_{t-1},$$

with:

$$\Delta Y_t = Y_t - Y_{t-1}$$

and:

$$q_t = (Q_t - Q_{t-1})/Q_{t-1}$$

$$p_t = (P_t - P_{t-1})/P_{t-1}$$
(3.2)

Hence q_t and p_t are the rates of growth in conventional production factors and social overhead capital and β_t and γ_t are the respective elasticities of output with respect to these inputs. Such a discrete approximation of a model with a continuous time trajectory is usually valid within the range for which the structure of the economic system is stable, and within this range the system will exhibit a non-cyclical growth. This self-sustained growth path may be drawing to a close because of either external causes (e.g., scarcity of production factors or lack of demand) or internal forces (e.g. emergence of dis-economies of scale and scope leading to negative marginal products).

External factors may drive the system toward an upper limit set by the new constraints concerned. **Internal** factors may lead to perturbations and qualitative changes in systemic behaviour. Suppose for instance, a capacity constraint caused by too high a concentration of capital in a production system. Then each additional increase in productive capital will have a negative impact on output. This implies that the production elasticity has become a negative time-dependent variable. In other words, beyond the capacity limit Y_{\max} an auxiliary relationship reflecting a negative marginal product of conventional production factors may be assumed, for instance, of the following form:

$$\beta_t = \beta^* (Y_{\max} - \kappa Y_{t-1})/Y_{\max} \quad (3.3)$$

In practice the economy may not move beyond Y_{\max} , but equation (3.3) shows that as it approaches Y_{\max} , the elasticity of output with respect to conventional production factors decreases at a rate of $-\kappa\beta^*/Y_{\max}$. Substitution of (3.3) into (3.2) leads to the following adjusted dynamic production function:

$$\Delta Y_t = v_t (Y_{\max} - \kappa Y_{t-1}) Y_{t-1} / Y_{\max} + \gamma p_t Y_{t-1} \quad (3.4)$$

where $v_t = \beta^* q_t$. This is seemingly a fairly simple non-stochastic dynamic relationship, but it can be shown that this equation is able to generate - under certain conditions - **unstable and even erratic behaviour** leading to a-periodic fluctuations. It is evident that the evolution of v_t itself is likely to be endogenous. The accumulation of capital and human capital, for example, is a function of the level of real income. Hence $\Delta q_t = a(Y_t)$ and, thus, $\Delta v_t = \beta^* \Delta q_t = \beta^* a(Y_t)$. Similarly, the capacity limit Y_{\max} may be affected by investments in social overhead capital, i.e. $\Delta Y_{\max} = b(q_t)$.

Combining this with equation (3.4), the following dynamic system emerges

$$\Delta Y_t = v_t (1 - \kappa Y_{t-1}/Y_{\max}) Y_{t-1} + \gamma p_t Y_{t-1} \quad (3.5)$$

$$\begin{aligned} \Delta v_t &= \beta^* a(Y_t) \\ \Delta Y_{\max} &= b(q_t) \end{aligned}$$

It is noteworthy that system (3.5) is an example of a Lotka-Volterra type model, which has often been used in recent years to model predator-prey relationships in population dynamics (see also Goh and Jennings, 1977; Jeffries 1979; Pimm 1982; and Wilson 1981). However, for the sake of expository purposes we will abstract here from the positive feedback from output to inputs and return to that issue in a model of endogenous input accumulation and technological change in the next section. Hence here we assume that the rates of change in conventional production factors and social overhead capital are both exogenously given. The endogeneity of Y_{\max} does also not affect the property of the model we will focus on and hence for the sake of simplicity we will assume that Y_{\max} is fixed over the period of interest. Hence (3.5) reduces to:

$$\Delta Y_t = \beta^* q (1 - \kappa Y_{t-1}/Y_{\max}) Y_{t-1} + \gamma p Y_{t-1} \quad (3.6)$$

The model represented by equation (3.6) has a similar structure to non-linear difference equations studied by May (1974), Li and Yorke (1975) and Yorke and Yorke (1975). Applications in a geographical setting can be found in Brouwer and Nijkamp (1985), Dendrinos and Müllally (1983, 1984) and Nijkamp and Reggiani (1989) among others.

Equation (3.6) is a standard equation from population dynamics. It should be noted that logistic evolutionary patterns may also be approximated by a (slightly more flexible) Ricker curve (see May 1974). In that case, the exponential specification precludes the generation of negative values for the Y variables in simulation experiments, a situation that may emerge in relation to equation (3.6).

It can be easily seen that there are two steady-states, $\bar{Y} = 0$ and $\bar{Y} = (1 + \gamma p / \beta q) / \kappa$. However, the stability of the system out of the steady-state equilibria is a complex issue.

Model (3.6) has some very unusual properties. On the basis of numerical experiments, it was demonstrated by May that this model may exhibit a remarkable spectrum of dynamical behaviour, such as stable equilibrium points, stable cyclic oscillations, stable cycles, and chaotic regimes with a-periodic but bounded fluctuations. Two major elements determine the stability properties of (3.6), viz. the **initial values** of Y_t and the tuning parameters (in our case β^* and q) which affect the **growth rate** for the economic system. Simulation experiments indicated that especially the tuning parameters have a major impact on the emergence of cyclic or a-periodic fluctuations. May has demonstrated that a stable equilibrium may emerge if $0 \leq \beta^* q \leq 2$ (and $p = 0$); otherwise stable cyclic and unstable fluctuations may be generated. Li and Yorke (1975) have later developed a set of sufficient conditions for the emergence of chaotic behaviour for general continuous difference equations.

Clearly, in our discrete model the potential chaotic behaviour depends on the value of β^* and q. It is easily seen from (3.6) that our dynamic model is essentially nothing else but an expression for the growth rate of output generated by the new technological conditions reflected in the production elasticity β^* . Usually such a relative change is positive but less than or equal to 1. It is thus plausible to stipulate that only in case of drastic or structural changes β^* is larger than 1. Similarly, conventional production factors grow at a rate of a few percent per annum. Hence even if the degree of homogeneity of the Cobb-Douglas production function would be higher than 1, $\beta^* q$ would be relatively small, as in case of a normal evolutionary pattern the relative changes in production factors will not be excessively high. Thus in case of incremental changes it is clear that $\beta^* q \leq 1$, so that then a stable equilibrium is ensured; otherwise many alternative evolutionary patterns of the system concerned may emerge. Consequently, the conclusion may be drawn that - due to the presence of a capacity limit

Y_{\max} - an economy might in principle exhibit a wide variety of dynamical or even cyclical growth patterns, although in this case the emergence of chaos does not seem to be very likely if we consider only short-term small changes.

The variety of behaviour generated by equation (3.6) can be easily demonstrated by means of two simple simulation experiments. In the first experiment there are in the absence of capacity constraints, constant returns to scale with respect to conventional production factors, i.e. $\beta^* = 1$. These production factors are assumed to grow at a rate of 5 percent per annum ($q = 0.05$). The elasticity of output with respect to social overhead capital is set at $\gamma = 0.3$, while this input grows at a rate of 3 percent ($p = 0.03$). Finally, output is scaled such that $Y_{\max} = 1$, $Y_0 = 0.1$ and the parameter representing the congestion and other decreasing returns effects $\kappa = 1.4$. In this case, Figure 2 shows that output growth follows a logistic curve with a long-run static equilibrium at $\bar{Y} = (1 + \gamma p / \beta q) / \kappa = (1 + 0.3 \times 0.03 / (1 \times 0.05)) / 1.4 = 0.843$.

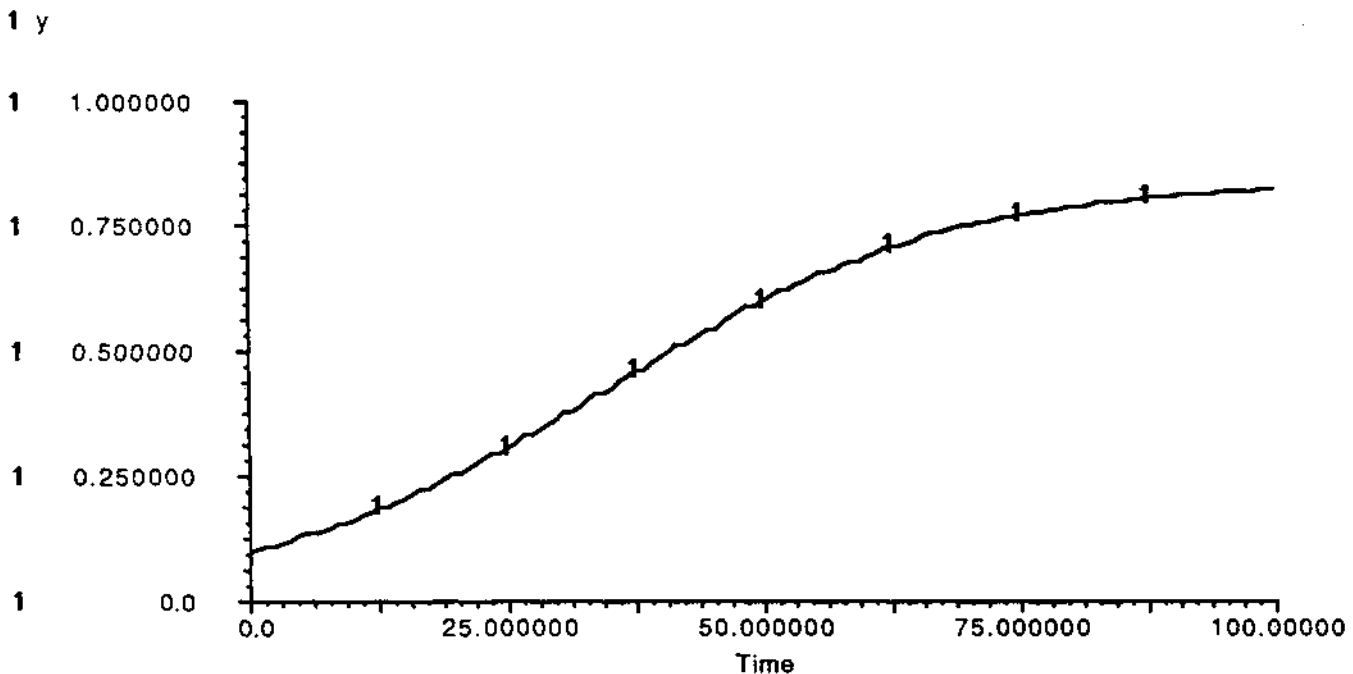


Figure 2. Results of a simulation run with growth converging to a stationary state.

In the second simulation, we change the time window by considering discrete shocks to productivity occurring every decade. Thus, the time index t refers now to 10 year periods. We also consider much faster

growth than in the first case: conventional production factors grow at a rate of 14 percent per annum (e.g. due to a rapid influx of migration and foreign capital). Thus, over the decade the compound growth rate is $q = (1.14)^{10} - 1 = 2.7$. However, social overhead capital continues to grow at a rate of 3 percent p.a., i.e. $p = (1.03)^{10} - 1 = 0.34$. As before, the steady-state equilibrium can be easily computed, $\bar{Y} = 0.741$. but this equilibrium is now highly unstable. Figure 3 shows that the economy now exhibits wild fluctuations.

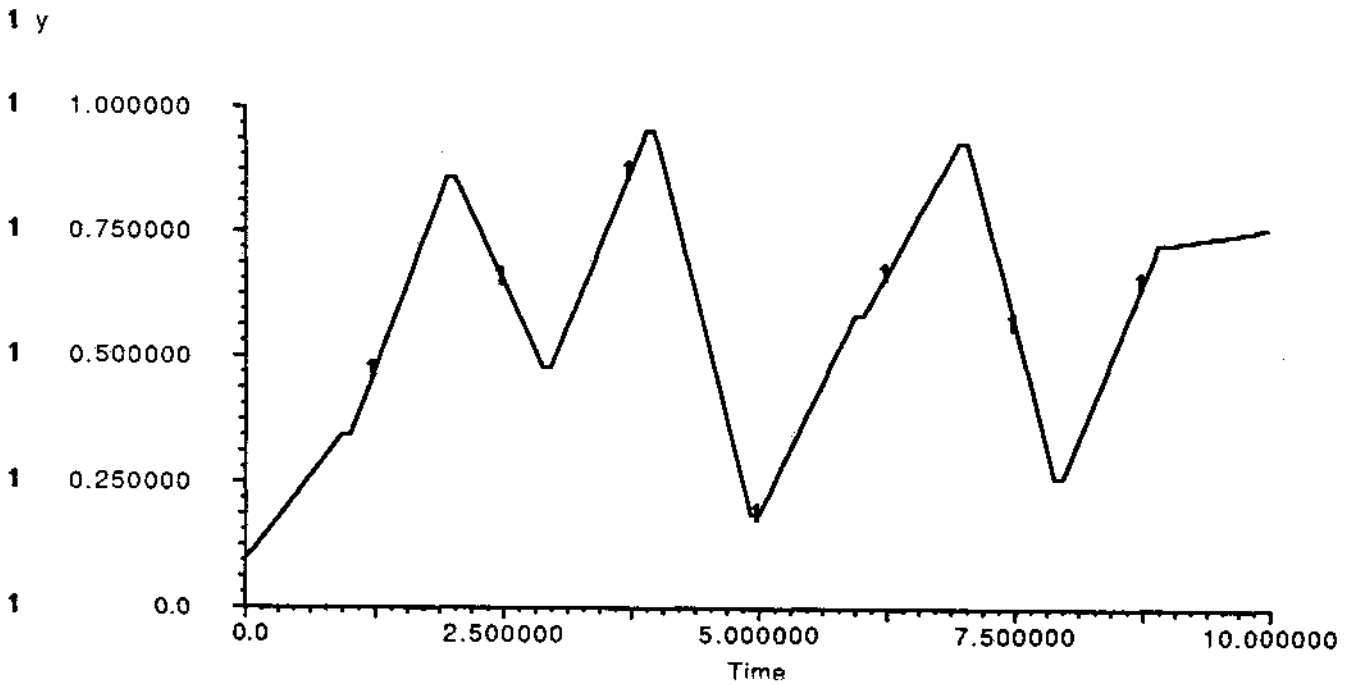


Figure 3 Results of a simulation run with persistently unstable growth.

It is well known that the outcome of the second simulation run is entirely the consequence of the specification of the model in difference equation form and the choice of the unit of time. In differential equation form, model (3.6) would exhibit global convergence to the long-run steady-state (see also May, 1974). However, economic phenomena often exhibit discontinuities and discrete lags. In this case, a difference equation specification would be quite plausible. Moreover, if the non-linear model contains three or more interacting variables (as is the case in system (3.5)) it may exhibit chaotic patterns and strange attractor sets (rather than a single equilibrium point) even in differential equation form. A well known example is provided by the

three-equation Rössler model (1976), which has been applied, for example, in a generic management model by Rasmussen and Mosekilde (1988).

In general, the plausibility of the model outcomes would depend on the specification of the model and rigorous empirical scrutiny of its parameters. It must be noted that key parameters (such as the tuning parameter in the May model), which define the qualitative dynamics of the system, may themselves be endogenous and push the system from a chaotic regime to a periodic cycle or stable equilibrium. In this case systems tend to exhibit self-organizing behaviour (see e.g. Radzicki, 1990).

For example, in our case there is a difference with respect to May's model. In May's model, v is a constant, whereas in our case v is endogenously determined by the evolution of our economic system (see equation (3.5)). This has clearly an effect on the growth trajectory, but - given the conditions on v_t - this does not affect the main conclusions regarding the stability of the system, although it has to be realized that drastic changes in any period are likely to generate perturbations in the next period. Since the growth rate v_t is not necessarily a constant, it may become an endogenous variable which may be used as an instrument variable in order to generate a more stable growth path, or to maximise a welfare criterion.

In the latter case an optimal control model emerges. It has been recently discovered that control problems in which there are at least two state variables may generate endogenous and persisting cycles (Feichtinger, 1990) so that even in competitive markets with rational economic agents the system may exhibit persisting, but bounded, fluctuations.

4. An Analysis of R&D Impacts in Constrained Development

Until now we have not explicitly considered the process by which productive inputs accumulate; only their productivity in the presence of rapid growth and capacity constraints was analysed. In this section we will present a dynamic model of the impact of endogenous R&D in such a constrained economy. Particular attention will again be given to stability properties under conditions of diseconomies of scale.

Research and development (R&D) has become a focal point in current evolutionary economics (cf. Kamien and Schwartz 1982, Nelson and Winter 1982, Nijkamp 1985, and Scherer 1980). R&D decisions lead, like investment in conventional capital goods, to an interesting intertemporal

allocation problem: more R&D expenditures may generate a rise in long-run productivity and profitability, but reduces short-run consumption, and vice versa. This choice problem for capital formation has been extensively studied in traditional economic growth theory, both for economies on a steady state growth path ('golden rule of accumulation') and in the framework of an intertemporal welfare optimisation problem by means of optimal control (see e.g. Ramanathan, 1982).

In the last few years there has been a remarkable revival of interest in economic growth theory (see Barro and Romer, 1990; and Ehrlich, 1990, for overviews). Of particular importance is the role that endogenous technological change can have in the process of development. Such technological change can be the result of human capital accumulation, learning by doing, R&D, innovation diffusion and other forms of spillover effects and spatial interaction (Nijkamp and Poot, 1991).

In general, an important question emerges in relation to R&D and economic growth. The growth path of the economy in an integrated consumption, production, investment and R&D system may be constrained, when the system is facing **capacity limits** (e.g., congestion, other diseconomies of scale, or depletion of exhaustible resources). In Nijkamp et al (1991) the long-run evolutionary path of such an economy was analyzed by means of a dynamic (discrete-time) model incorporating the generation of technological change under conditions of diseconomies of scale. In this section some elements of their approach will be taken up again. It will be shown that a constrained dynamic economic system may generate a wide range of dynamic behaviour, including - for certain parameters - chaotic evolution. We assume that the production in the economy under consideration can be described by means of the following simple production function:

$$Y_t = \epsilon_t K_t \quad , \quad (4.1)$$

with K_t the installed capital stock at the beginning period t and ϵ_t a time-dependent technological coefficient representing average capital productivity during the period $(t, t+1)$. It should be noted that this linearity assumption is not as restrictive as it seems, since in a sense we may consider (4.1) an identity in which ϵ_t includes all other relevant factors (labour, land, social overhead capital, R&D) which influence capital productivity. Consequently, the elasticity of substitution between capital and other production factors is not assumed to

be zero; the time trajectory of ϵ can incorporate both substitution between production factors as well as technological change.

Capital accumulation can be described by means of the following standard expression:

$$K_{t+1} = (1-\delta) K_t + I_t \quad (4.2)$$

with I_t gross investment during period $(t, t+1)$ and δ the rate of physical depreciation of capital. Now we assume the following simple investment function:

$$I_t = \sigma_1 Y_t \quad (4.3)$$

with σ_1 the average savings rate (assuming the existence of equilibrium between savings and capital increase). The value of σ_1 will be the outcome of an intertemporal optimisation problem of economic agents in a competitive economy. On a long-run steady-state growth path, v_1 would be constant and its value a function of inter alia the discount rate, the technology, the welfare function and population growth.

We take for granted that the current production efficiency can be increased through R&D embodied in the production technology. This requires a change in the production function, as R&D investments will increase efficiency due to a change in the capital coefficient (see Baumol and Wolff, 1984; Mansfield, 1980; Nelson, 1981). In other words, a new 'technological regime' (cf. Nelson and Winter, 1982) requires R&D expenditures with a positive impact on the production efficiency parameter ϵ_t . In our model this effect will be indicated by a parameter ν_t , which measures the impact on capital productivity as a result of an additional unit of R&D. This leads to the following equation:

$$\Delta \epsilon_t = \epsilon_{t+1} - \epsilon_t = \nu_t R_t \quad (4.4)$$

where R_t represents the R&D investments during period $(t, t+1)$ and ν_t the R&D impact parameter for the capital coefficient. Next we may introduce a relationship for R_t , which defines the savings rate for R&D as a proportion of income:

$$R_t = \sigma_2 Y_t \quad (4.5)$$

Again, the value of σ_2 can be the outcome of an intertemporal optimisation problem. In any period, it is obvious that the amount of output available for consumption C_t is given by:

$$C_t = (1 - \sigma_1 - \sigma_2) Y_t \quad , \quad (4.6)$$

Next, substitution of (4.5) into (4.4) leads to the following result:

$$\Delta \epsilon_t = \nu_t \sigma_2 Y_t \quad (4.7)$$

If ν_t were constant over time, capital accumulation would generate ever-increasing growth in output and capital productivity. This highly unlikely outcome suggests that ν_t is likely to decrease when output increases. In other words, the marginal efficiency of R&D declines when production increases. Under a given 'technological regime', ultimately a 'saturation' level of output $Y_{t,\max}$ is likely to exist at which additional R&D has no longer an impact on productivity. Such a saturation level (ceiling) may arise from capacity limits (technological, social, economic) and reflects - for a given production technology - a 'limits to growth' phenomenon, stemming from diseconomies of scale and scope, as in the previous section. Arguments in favour of the assumption of a decreasing productivity of R&D in case of more mature economic conditions can also be found in Ayres (1987) and Metcalfe (1981) among others.

In view of the above observations it is now clear that $\nu_t = 0$ when $Y_t \geq Y_{t,\max}$. Naturally, these limits to growth themselves may be subject to change, so that $Y_{t,\max}$ may increase with time and - as prevailing bottlenecks are overcome - new R&D may again have a positive effect on productivity. Thus, the following specification for an adjusted (i.e., time-dependent) R&D impact parameter seems plausible:

$$\nu_t = \max \{ \nu^* (1 - Y_t / Y_{t,\max}), 0 \} \quad (4.8)$$

Furthermore, it is plausible that not only would R&D expenditure become ineffective if output expands beyond $Y_{t,\max}$, but it may also be expected that diseconomies of scale and scope set in which reduce capital productivity. The previous remarks indicate that instead of (4.4) we may now have the following simple relationship for the change in capital productivity:

$$\Delta \epsilon_t = \nu_t R_t - \mu_t Y_t \quad (4.9)$$

in which μ_t measures the effect of diseconomies on productivity when output exceeds $Y_{t,\max}$, so that:

$$\mu_t = \max(\mu^* (Y_t/Y_{t,\max} - 1), 0) \quad (4.10)$$

Assuming for simplicity that $Y_{t,\max}$ grows at the exogenous rate n and substituting (4.10) and (4.8) into (4.4), the motion in the system can now be described by the following set of non-linear difference equations;

$$K_{t+1} = (1-\delta) K_t + \sigma_1 \epsilon_t K_t \quad (4.11)$$

$$\begin{aligned} \epsilon_{t+1} = & \epsilon_t + \{\sigma_2 \nu^* \max(1 - Y_t/Y_{t,\max}, 0) - \\ & - \mu^* \max(Y_t/Y_{t,\max} - 1, 0)\} \epsilon_t K_t \end{aligned}$$

$$Y_{t+1} = \epsilon_{t+1} K_{t+1}$$

$$Y_{t+1, \max} = (1+n)^t Y_{0, \max}$$

In view of the non-linear properties of this model, it is clear that for any given initialisation $(K_0, \epsilon_0, Y_0, Y_{0,\max})$ system (4.11) can exhibit a wide range of time trajectories dependent on parameter values.

Some results based on simulation experiments are illustrated in Figures 4 and 5. Figure 4 is based on the assumption that $K_0=1000$ and the capital-output ratio equals 5; hence $\epsilon_0=0.2$ and $Y_0=200$. The saving ratio is 20 percent; 2 percent of the capital stock is assumed to become obsolete each period and 2 percent of income is spent on R&D, so that $\sigma_1=0.20$ and $\delta=\sigma_2=0.02$. The sustainable output capacity $Y_{0,\max}=1000$ and grows at 1 percent. Moreover, $\mu^*=0.0001$ and $\nu^*=0.001$. Since $5\sigma_2\nu^*=\mu^*$, the productivity response is five times as elastic when $Y_t > Y_{t,\max}$ than when $Y_t < Y_{t,\max}$, and of opposite sign.

Figure 4 shows that growth in the system is - under these conditions - in initial periods **accelerating**. However, the growth rate of capital productivity reaches a maximum at $t=25$ and subsequently declines, until Y_t reaches the saturation level $Y_{t,\max}$ at $t=37$. At this point, the growth rate of capital accumulation reaches a maximum. Beyond $t=37$, Y_t will remain above $Y_{t,\max}$ but it will converge to the latter

value. Consequently, capital productivity becomes constant at a rate of $(n+\delta)/\sigma_1=0.15$, whilst capital and output grow at a steady-state rate of 1 percent.

Next, we assume that in Figure 5 all parameters are the same as in Figure 4, but μ^* has been increased to five times its former value. Consequently, the effect of diseconomies is now sufficiently strong to push \dot{Y}_t at times below $Y_{t,max}$ so that **growth cycles** are generated with a variable **periodicity** but with decreasing amplitude. The system eventually converges again to a steady-state growth of 1 percent. Thus stringent diseconomies cause the system to be more chaotic.

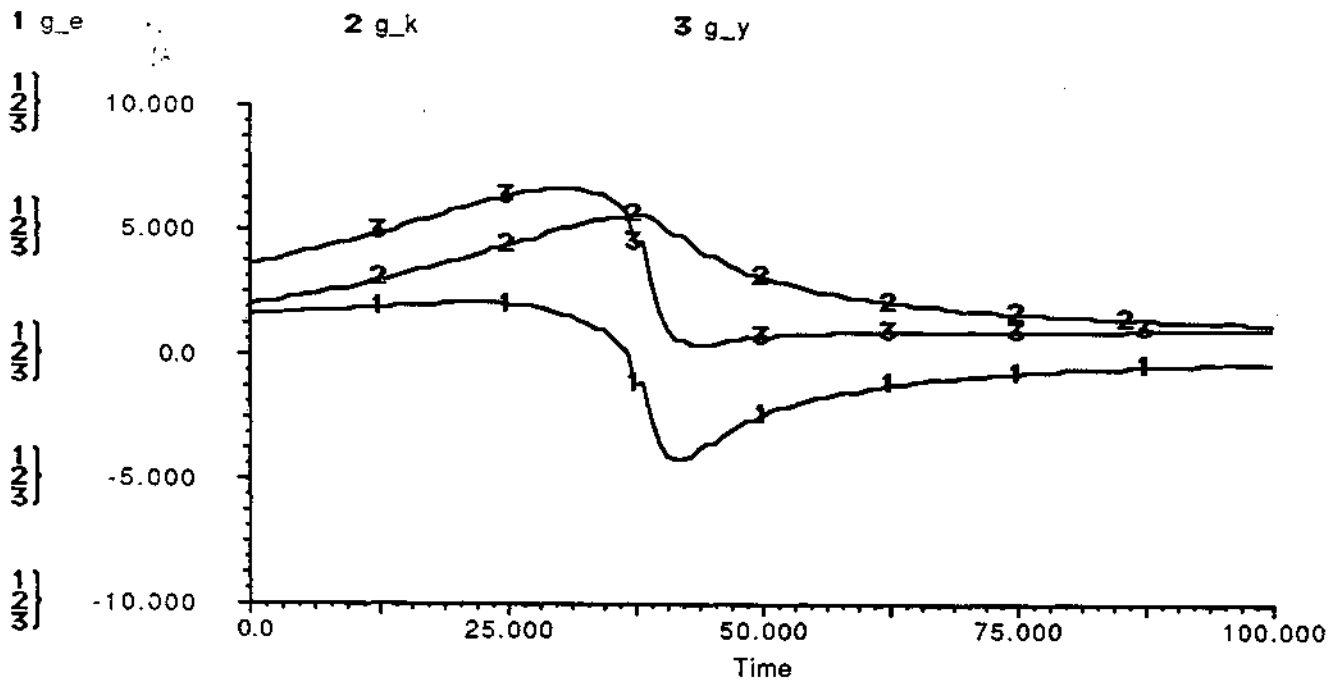


Figure 4. Growth converging to a steady state.

Legend: 1 : growth in capital productivity
 2 : capital growth rate
 3 : income growth rate

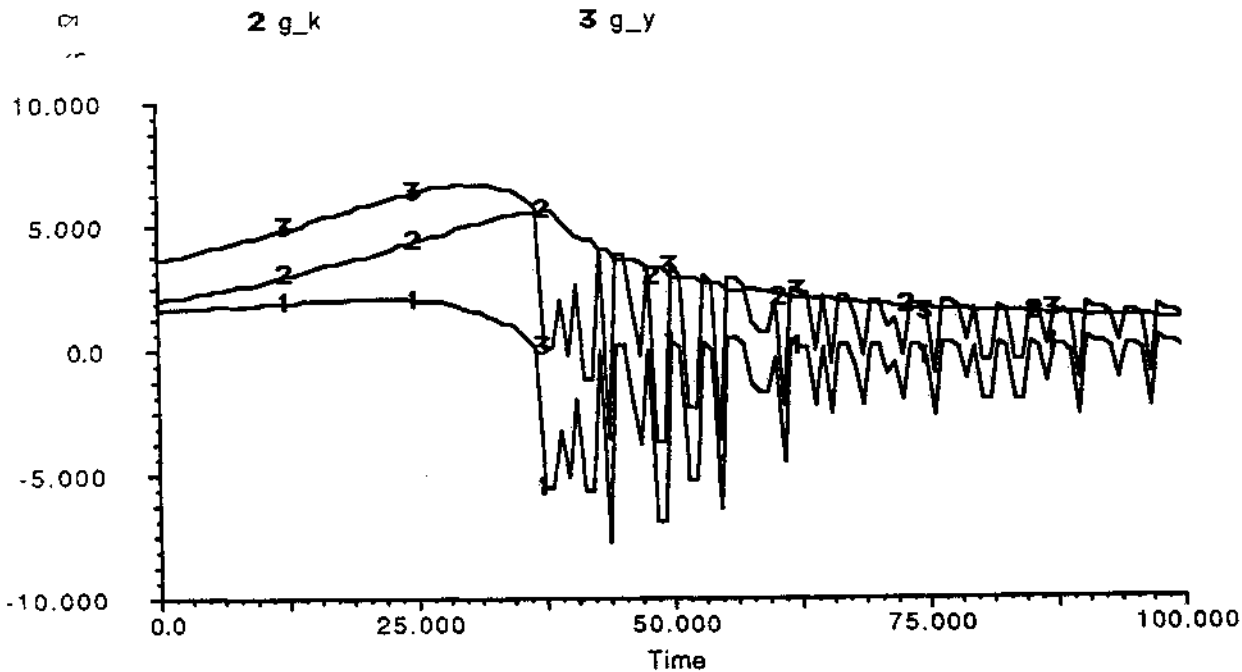


Figure 5. Growth cycles generated by strong external diseconomies.

The earlier noted sensitivity of models which exhibit chaotic behaviour to parameter values or initial conditions can be easily demonstrated by means of the model discussed in this section. Figure 6 duplicates the fluctuations in the income growth rate displayed in Figure 5 but the growth rates of capital and capital productivity have been deleted for clarity, while the focus is on period 60 to 100 only. Figure 6 shows the outcome of simulation with exactly the same model, but with parameter μ increased by 1 percent (i.e. from 0.0005 to 0.000505). This very small change in the parameter value generates nonetheless fluctuations in the growth rate of income which, for a large proportion of the time, are very different from those for the earlier simulation. Hence even if the model which generated Figure 6 would be known perfectly, except for the exact value of one of the parameters, such as μ , it would still be impossible to forecast the level of income for any period but the very near future.

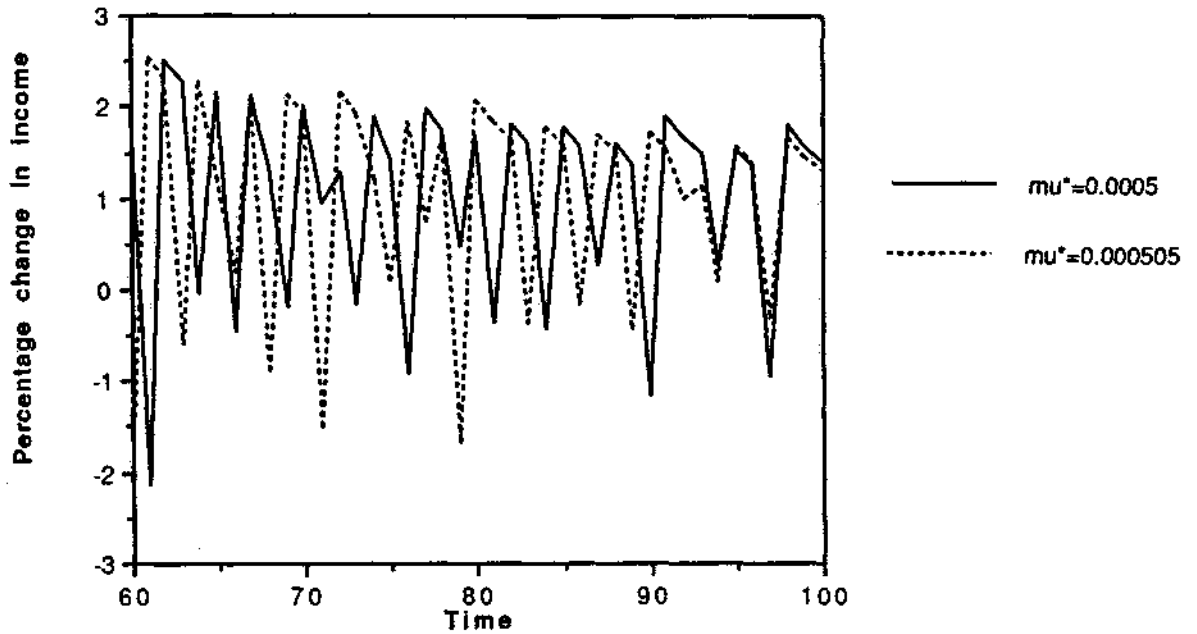


Figure 6. The high sensitivity of the non-linear growth model to parameter changes.

Similarly, a small change in the initialisation of the system can generate after some time drastic qualitative and quantitative changes in the time trajectories of the variables of the system. It should be noted that such a sensitive dependence to initial conditions and parameter value applies also to unstable linear systems, but in the non-linear case the resulting time trajectories may remain bounded, while in the linear case any divergence will be monotonic.

5. Retrospect

The previous experiments have demonstrated that even conventional economic models may exhibit irregular behaviour in case of high growth rates or strict limits to growth. In various cases this may be beyond plausible empirical values of an economic system, so that chaos in a real-world system is less likely to emerge. However, in case of rapid transitions or sudden adjustments such chaos patterns may temporarily emerge.

If instead of a simulation experiment, one would have to use models of the above nature as normative policy models, it would be necessary to introduce an appropriate objective (or welfare) function encompassing a trade-off between relevant welfare arguments. A dynamic programming or

optimal control formulation would then be desirable (see Kendrick, 1981, and Nijkamp and Reggiani, 1988, 1989). Such a constrained dynamic optimization might in principle reduce chaotic fluctuations inherent in the nonlinear dynamics of the growth model and provide a self-organising stabiliser for the system.

Another point concerns the specification of non-linear dynamic models. It may be important to stress that the foundations of specifying an economic model have to be firmly rooted in economic theory, as otherwise we run the danger of ad hoc theorizing and econometric misspecification, which may generate chaotic behaviour that is not based on plausible economic grounds.

However, it is extremely unlikely that purely deterministic models will explain the outcomes of interaction between economic agents. The central question for future research is whether observed volatility is the result of linear structural dynamics combined with exogenous shocks and stochastic noise due to measurement error, or alternatively, deterministic non-linear dynamics in which stochastic terms play a minor role. Since the number of data points in economic phenomena is so much smaller than in, for example, the experimental sciences in which non-linear dynamics is becoming very popular, non-linear determinism and the presence of a large number of exogenous shocks are observationally equivalent.

Finally, it is important to call attention to the fact that in various cases a system is not chaotic as a whole, but has only a few 'niches' (modules or equations) which under certain conditions may exhibit chaotic behaviour. The question whether chaotic behaviour of a small sub-system will be dampened by the dominance of another and otherwise stable system, or whether it will exert an explosive influence upon a whole system needs further investigation.

Although non-linear models for economic development may provide an interesting explanatory framework for the rise and decline of regions and nations in a dynamic (sometimes chaotic) context, it is also evident from the above experiments that the 'economics of chaos' desperately needs rigorous empirical research work. The challenge is to build theoretical models which combine small amounts of randomness with nonlinearities and succeed in generating data that replicate real economic and financial time series.

References

Albin, P.S., Microeconomic Foundations of Cyclical Irregularities or Chaos, Mathematical Social Sciences, vol. 13, 1987, pp. 185-214.

Allen, A., The New Zealand Sharemarket Gross Index and Chaos: Some preliminary evidence, Discussion Paper in Money and Finance, Faculty of Commerce and Administration, Victoria University, 1989.

Allen, P.M. and M. Sanglier, A Dynamic Model of Growth in a Central Place System, Geographical Analysis, vol. 11, 1979, pp. 256-272.

Andersen, D.F. (ed.), Chaos in System Dynamic Models: Special Issue System Dynamics Review, vol. 4, no. 1/2, 1988.

Ayres, R.U., Barriers and Breakthroughs: An "Expanding Frontiers" Model of the Technology-Industry Life Cycle, Research Paper, IIASA, Laxenburg, 1987.

Balducci, R., G. Candela and G. Ricci, A Generalization of R. Goodwin's Model with Rational Behaviour of Economic Agents, Nonlinear Models of Fluctuating Growth, (R.M. Goodwin, M. Krüger and A. Vercelli, eds.), Springer-Verlag, Berlin, 1987, pp. 47-66.

Barnett, W. and P. Chen, The Aggregation-theoretic Monetary Aggregates Are Chaotic and Have Strange Attractors, Dynamic Econometric Modelling, (W. Barnett, E. Berndt and H. White, eds.), Cambridge University Press, 1988, pp. 199-245.

Barnett, W., J. Geveke, and K. Shell (eds.), Economic Complexity: Chaos, Sunspots, Bubbles and Nonlinearity, Cambridge University Press, Cambridge, 1990.

Barro, R.J. and P.M. Romer (1990), Economic Growth, NBER Reporter, National Bureau of Economic Research, Fall 1990, pp. 1-5.

Batty, M., Cities as Fractals: Simulating Growth and Form, Fractals and Chaos (A.J. Crilly, R.A. Earnshaw, and H. Jones, eds.), Springer Verlag, Berlin, 1991, pp. 43-70.

Baumol, W.J. and J. Benhabib, Chaos: Significance, Mechanism, and Economic Applications, Journal of Economic Perspectives, vol. 3, no. 1, 1989, pp. 77-105.

Baumol, W.J. and R.E. Quandt, Chaos Models and their Implications for Forecasting, Eastern Economic Journal, vol. 11, no.1, 1985, pp. 3-15.

Baumol, W.J., and E.N. Wolff, Feedback from Productivity Growth in R&D, Scandinavian Journal of Economics, vol. 85, 1983, pp. 145-157.

Baumol, W.J., and E.N. Wolff, Feedback Models: R&D Information, and Productivity Growth, Communication and Information Economics, (M. Jussawalla and H. Ebenfield, eds.), North-Holland Publ. Co., Amsterdam, 1984, pp. 73-92.

Benhabib, J. and R.H. Day, Rational Choice and Erratic Behaviour, Review of Economic Studies, vol. 48, 1981, pp. 459-471.

Benhabib, J. and R.H. Day, A Characterization of Erratic Dynamics in the Overlapping Generations Model, Journal of Economic Dynamics and Control, vol. 4, 1982, pp. 37-55.

Blommestein, H.J., Eliminative Econometrics, Ph.D. Diss., Department of Economics, Free University, Amsterdam, 1986.

Boldrin, M., Persistent Oscillations and Chaos in Dynamic Economic Models: Notes for a Survey, SFI Studies in the Science of Complexity, Addison-Wesley Publ. Co., 1988, pp. 49-75.

Boldrin, M., and M. Woodford, Equilibrium Models Displaying Endogenous Fluctuations and Chaos: A Survey, Journal of Monetary Economics, vol. 25, 1990, pp. 189-222.

Brock, W.A., Chaos and Complexity in Economic and Financial Science, in G.M. von Furstenberg (ed.), Acting under Uncertainty: Multidisciplinary Conceptions, Kluwer, Norwell, MA, 1989.

Brock, W.A., Applications of Nonlinear Science Statistical Inference Theory to Finance and Economics. Proceedings of the Spring Seminar of the Institute for Quantitative Research in Finance, Innisbrook, Florida, 1988.

Brock, W.A., Distinguishing Random and Deterministic Systems, Journal of Economic Theory, vol. 40, 1986, pp. 168-195.

Brock, W.A., Dechert, W.D. and J.A. Scheinkman, A Test for Independence Based on the Correlation Dimension, SSRI Working Paper No. 8702, Department of Economics, University of Wisconsin at Madison, 1987.

Brock, W.A. and C.L. Sayers, Is the Business Cycle Characterized by Deterministic Chaos? Journal of Monetary Economics, vol. 22, 1988, pp. 71-90.

Brouwer, F. and P. Nijkamp, Qualitative Structure Analysis of Complex Systems, Measuring the Unmeasurable; Analysis of Qualitative Spatial Data (P. Nijkamp, H. Leitner and N. Wrigley, eds.), Martinus Nijhoff, The Hague, 1985, pp. 509-532.

Casetti, E., Technological Progress, Exploitation and Spatial Economic Growth: A Catastrophe Model, Dynamic Spatial Models (D.A. Griffith and R. MacKinnon, eds.), Sijthoff and Noordhoff, Alphen aan de Rijn, 1981, pp. 215-277.

Chen, P., Empirical and Theoretical Evidence of Economic Chaos, System Dynamics Review, vol. 4, no. 1, 1988, pp. 81-108.

Chiarella, C., The Cobweb Model: Its Instability and the Onset of Chaos, Economic Modelling, vol. 5, no. 4, 1988, pp. 377-384.

Crilly, A.J., R.A. Earnshaw, and H. Jones (eds.), Fractals and Chaos, Springer Verlag, Berlin, 1990.

Crutchfield et al., Fluctuations and Simple Chaotic Dynamics, Physics Reports, 1982, vol. 92, pp. 45-82.

Dana, R.A. and L. Montrucchio, Dynamic Complexity in Duopoly Games, Journal of Economic Theory, vol. 40, 1986, pp. 40-56.

- Day, R.H., Irregular Growth Cycles, American Economic Review, vol. 72, 1982, pp. 406-414.
- Dendrinos, D.S. (ed.), Dynamic Non-Linear Theory and General Urban/Regional Systems, School of Architecture and Urban Design, Lawrence, Kansas, 1981.
- Dendrinos, D.S., Turbulence and Fundamental Urban/Regional Dynamics, Paper presented at the American Association of Geographers, Washington D.C., April 1984.
- Dendrinos, D.S., On the Incongruous Spatial Employment Dynamics, Technological Change, Employment and Spatial Dynamics, (P. Nijkamp, ed.), Springer-Verlag, Berlin, 1986, pp. 321-339.
- Dendrinos, D.S. and H. Mullally, Empirical Evidence of Volterra-Lotka Dynamics in United States Metropolitan Areas: 1940-1977, Evolving Geographical Structures (D.A. Griffith and T. Lea, eds.), Martinus Nijhoff, The Hague, 1983, pp. 170-195.
- Dendrinos, D.S. and H. Mullally, Urban Evolution, Oxford University Press, Cambridge, 1984.
- Dendrinos, D.S. and M. Sonis, The Onset of Turbulence in Discrete Relative Multiple Spatial Dynamics, Applied Mathematics and Computation, vol. 22, 1987, pp. 25-44.
- Deneckere, R. and S. Pelikan, Competitive Chaos, Journal of Economic Theory, vol. 40, 1986, pp. 13-25.
- Devaney, R.L., Chaotic Dynamical Systems, Benjamin Cummings Publ. Co., Menlo Park, CA, 1986.
- Ehrlich, I., The Problem of Development: Introduction, Journal of Political Economy, vol. 98, 1990, no. 5.2, pp. S1-S11.
- Fama, E., Foundations of Finance, Basic Books, New York, 1976.
- Feichtinger, G., Optimal Periodic Policies in Dynamic Economic Systems, University of Technology, Vienna, Austria, 1990, mimeo.
- Finkenstädt, B. and P. Kuhbier, Chaotic Dynamics in Agricultural Markets, Free University of Berlin, 1990, mimeo.
- Funke, M., A Generalized Goodwin Model Incorporating Technical Progress and Variable Prices, Economic Notes, vol. 2, 1987, pp. 36-47.
- Goh, B.S. and L.S. Jennings, Feasibility and Stability in Randomly Assembled Lotka-Volterra Models, Ecological Modelling, vol. 3, no. 1, 1977, pp. 63-71.
- Goodwin, R.M., Essays in Economic Dynamics, Macmillan, London, 1982.
- Grandmont, J.M., On Endogenous Competitive Business Cycles, Econometrica, vol. 53, 1985, pp. 995-1046.
- Grandmont, J.M. (ed.), Special Issue Journal of Economic Theory, vol. 40, 1986.

Guckenheimer, J. and P. Holmes, Non-Linear Oscillations, Dynamical System and Bifurcation of Vector Fields, Springer Verlag, Berlin, 1983.

Guckenheimer, J., G. Oster and A. Ipatchi, The Dynamics of the Density Dependent Population Models, Journal of Mathematical Biology, vol. 4, 1977, pp. 101-147.

Haag, G. and W. Weidlich, A Nonlinear Dynamic Model for the Migration of Human Population, Evolving Geographical Structures (D.A. Griffith and T. Lea, eds.), Martinus Nijhoff, The Hague, 1983, pp. 24-61.

Hirschman, A.O., The Strategy of Economic Development, Yale University Press, New Haven, 1958.

Hommes, C.H. and H.E. Nusse, Does an Unstable Keynesian Unemployment Equilibrium in a Non-Walrasian Dynamic Macroeconomic Model Imply Chaos?, Scandinavian Journal of Economics, vol. 91, 1989, pp. 161-167.

Hommes, C.H., H.E. Nusse and A. Simonovits, Hicksian Cycles and Chaos in a Socialist Economy, Research Memorandum, Institute for Economic Research, State University, Groningen, 1990.

Hsieh, D., Testing for Nonlinear Dependence in Daily Foreign Exchange Rates, Journal of Business, vol. 62, 1989, pp. 339-368.

Jeffries, C., Qualitative Stability and Digraphs in Model Ecosystems, Ecology, vol. 55, no. 6, 1979, pp. 1415-1419.

Kamien, M. and N. Schwartz, Market Structure and Innovations, Cambridge University Press, Cambridge, 1982.

Kelsey, D., The Economics of Chaos or the Chaos of Economics, Oxford Economic Papers, 40, 1988, pp. 1-31.

Kendrick, D., Control Theory with Applications to Economics, Handbook of Mathematical Economics (K.J. Arrow and M.D. Intriligator, eds.), North-Holland Publ. Co., Amsterdam, 1981, pp. 111-158.

Kleinknecht, A., Innovation Patterns in Crisis and Prosperity: Schumpeter's Long Cycle Reconsidered, MacMillan, London, 1986.

Lasota, A. and M.C. Mackey, Probabilistic Properties of Deterministic Systems, Cambridge University Press, Cambridge, 1985.

LeBaron, B., Diagnosing and Simulating Some Asymmetries in Stock Return Volatility, Dept. of Economics, University of Wisconsin-Madison, mimeographed paper, 1989.

Li, T. and J.A. Yorke, Period Three Implies Chaos, American Mathematical Monthly, vol. 82, 1975, pp. 985-992.

Lichtenberg, A.J. and M.A. Lieberman, Regular and Stochastic Motion, Springer Verlag, Berlin, 1983.

Liosattos, P., Spatial Dynamics: Some Conceptual and Mathematical Issues, Environment & Planning A, vol. 12, 1980, pp. 1051-1071.

Lorenz, H.W., International Trade and the Possible Occurrence of Chaos, Economics Letters, 1987, vol. 23, pp. 135-138.

Lorenz, H.W., Nonlinear Dynamical Economics and Chaotic Motion, Springer Verlag, Berlin, 1989.

Lung, Y., Complexity and Spatial Dynamics Modelling: From Catastrophic Theory to Self-Organizing Process: a Review of Literature, The Annals of Regional Science, vol. 22, no. 2, 1988, pp. 81-111.

Mansfield, E., Industrial Research and Technological Innovation, W.W. Norton, New York, 1980.

May, R.M., Biological Populations with Non-overlapping Generations, Science, no. 186, 1974, pp. 645-647.

May, R.M., Simple Mathematical Models with Very Complicated Dynamics, Nature, 1976, vol. 261, pp. 459-467.

McKenzie, L.W. and S. Zamagni (eds.), Value and Capital: Fifty Years After, MacMillan, London, 1991.

Metcalf, J.S., Impulse and Diffusion in the Study of Technical Change, Futures, vol. 13, 1981, pp. 347-359.

Nelson R., Research and Productivity Growth and Productivity Differences, Journal of Economic Literature, vol. 8, 1981, pp. 1029-1064.

Nelson, R.D. and S.G. Winter, An Evolutionary Theory of Economic Change, Harvard University Press, Cambridge, 1982.

Nijkamp, P., Technological Change, Policy Response and Spatial Dynamics, Evolving Geographical Structures (D.A. Griffith and T. Lea, eds.), Martinus Nijhoff, The Hague, 1983, pp. 75-99.

Nijkamp, P., A Multidimensional Analysis of Regional Infrastructure and Economic Development, Regional and Industrial Development Theories, Models and Empirical Evidence (A.E. Andersson, W. Isard and T. Puu, eds.), North-Holland Publ. Co., Amsterdam, 1984, pp. 267-293.

Nijkamp, P., Employment, Technological Change and Spatial Dynamics, Springer Verlag, Berlin, 1985.

Nijkamp, P., Long-Term Economic Fluctuations: A Spatial View, Socio-Economic Planning Sciences, vol. 21, no. 3, 1987, pp. 189-197.

Nijkamp, P., Evolutionary Patterns of Urban Production Systems, Belgian Journal of Operations Research, Statistics and Computer Science, vol. 29, no. 2, 1989, pp. 23-40.

Nijkamp, P. and J. Poot, Endogenous Technological Change and Spatial Interdependence, Dept. of Economics, Free University, Amsterdam, 1991, mimeo.

Nijkamp, P., J. Poot, and J. Rouwendal, A Non-linear Dynamic Model of Spatial Economic Development and R&D Policy, Annals of Regional Science, 1991, forthcoming.

Nijkamp, P. and A. Reggiani, Dynamic Spatial Interaction Models: New Directions, Environment and Planning A, vol. 20, 1988, pp. 1449-1460.

Nijkamp, P. and A. Reggiani, Theory of Chaos: Relevance for Analysing Spatial Processes, Spatial Choices and Processes, (M.M. Fischer, P. Nijkamp, and Y. Papageorgiou, eds.), North-Holland Publ. Co., Amsterdam, 1989, pp. 49-80.

Nijkamp, P. and A. Reggiani, Logit Models and Chaotic Behaviour, Environment and Planning A, vol. 22, 1990, pp. 1455-1467.

Nijkamp, P., and A. Reggiani, Spatial Interaction, Evolution and Chaos, Springer Verlag, Berlin, 1992.

Nusse, H.E. and C.H. Hommes, Resolution of Chaos with Application to a Modified Samuelson Model, Journal of Economic Dynamics and Control, vol. 14, 1990, pp. 1-19.

Pimm, S.L., Food Webs, Chapman and Hall, London, 1982.

Pohjola, M.T., Stable and Chaotic Growth: the Dynamics of a Discrete Version of Goodwin's Growth Cycle Model, Zeitschrift für Nationalökonomie, vol. 41, 1981, pp. 27-38.

Priestley, M.B., Nonlinear and Non-Stationary Time Series Analysis, Academic Press, London, 1988.

Prigogine, I. and I. Stengers, Order out of Chaos, Fontana, London, 1985.

Prskawetz, A., Are Demographic Indicators Deterministic or Stochastic?. Institute for OR, Vienna Technical University, 1990, mimeo.

Radzicki, M.J., Institutional Dynamics, Deterministic Chaos and Self-Organizing Systems, Journal of Economic Issues, vol. 24, 1990, pp. 57-102.

Ramanathan, R., Introduction to the Theory of Economic Growth, Springer Verlag, Berlin, 1982.

Ramsey, J.P., C.L. Sayers and P. Rothman, The Statistical Properties of Dimension Calculations Using Small Data Sets: Some Economic Applications, International Economic Review, vol. 31, 1990, pp. 991-1020.

Rand, D., Exotic Phenomena in Games and Duopoly Models, Journal of Mathematical Economics, vol. 5, 1978, pp. 173-184.

Rasmussen, D.R. and E. Mosekilde, Bifurcations and Chaos in a Generic Management Model, European Journal of Operational Research, vol. 35, 1988, pp. 80-88.

Rasmussen, D.R., E. Mosekilde and J.D. Sterman, Bifurcations and Chaotic Behavior in a Simple Model of the Economic Long Wave, System Dynamics Review, vol. 1, 1985, pp. 92-110.

Reggiani, A., Spatial Interaction Models: New Directions, Ph.D. Dissertation, Dept. of Economics, Free University, Amsterdam, 1990.

Reiner, R., M. Munz, G. Haag and W. Weidlich, Chaotic Evolution of Migration Systems, Sistemi Urbani, vol. 2/3, 1986, pp. 285-308.

- Ricci, G., A Differential Game of Capitalism, Optimal Control Theory and Economic Analysis (G. Feichtinger, ed.), Elsevier, Amsterdam, 1985, pp. 633-643.
- Rosenberg, N. (1976) Perspective on Technology, Cambridge University Press, Cambridge, 1976.
- Rosser, J.B., From Catastrophe to Chaos: A General Theory of Economic Discontinuities, Kluwer, Dordrecht, 1991.
- Rössler, O.E., An Equation for Continuous Chaos, Physics Letters, 57A, July 1976, pp. 397-398.
- Rouwendal, J., and P. Nijkamp, Endogenous Production of R&D and Stable Economic Development, De Economist, vol. 137, no. 2, 1989, pp. 202-216.
- Scheinkman, J. A., Nonlinearities in Economic Dynamics, The Economic Journal, vol. 100, Conference 1990, pp. 33-48.
- Scheinkman, J. A. and B. LeBaron, Nonlinear Dynamics and Stock Returns, Journal of Business, July 1989, pp. 311-337.
- Scherer, F.M., Industrial Market Structure and Economic Performance, Rand McNally, Chicago, 1980.
- Schuster, H.G., Deterministic Chaos, Physik Verlag, Weinheim, FRG, 1984.
- Sonis, M., A Unified Theory of Innovation Diffusion, Dynamic Choice of Alternatives, Ecological Dynamics and Urban/Regional Growth and Decline, Ricerca Economica, vol. XL, 4, 1986, pp. 696-723.
- Sonis, M., Discrete Time Choice Models Arising from Innovation Diffusion Dynamics, Sistemi Urbani, no. 1, 1987, pp. 93-107.
- Sterman, J.D., A Behavioural Model of the Economic Long Wave, Journal of Economic Behaviour and Organization, vol. 5, 1985, pp. 17-53.
- Stewart, I., Does God Play Dice?, Basil Blackwell, Oxford, 1989.
- Stoneman, P., The Economic Analysis of Technological Change, Oxford University Press, Oxford, 1983.
- Stutzer, M., Chaotic Dynamics and Bifurcation in a Macro-model, Journal of Economic Dynamics and Control, vol. 2, 1980, pp. 253-276.
- Turner, J., Non-equilibrium Thermodynamics, Dissipative Structures and Self-Organization, Dissipative Structures and Spatio-temporal Organization Studies in Biomedical Research (G. Scott and J. MacMillan, eds.), Iowa State University Press, Iowa, 1980, pp. 13-52.
- White, R., Transition to Chaos with Increasing System Complexity: The Case of Regional Industrial Systems, Environment and Planning A, vol. 17, 1985, pp. 387 - 396.
- Wilson, A.G., Catastrophe Theory and Bifurcation, Croom Helm, London, 1981.
- Yorke, J.A., and E.D. Yorke, Chaotic Behaviour and Fluid Dynamics. Hydrodynamic Instabilities and the Transition to Turbulence (H.L. Swinney and J.P. Collub, eds.), Springer Verlag, New York, 1981, pp. 112-128.