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A NEW HEURISTIC FOR THE OVERFLOW PROBABILITY IN FINITE-BUFFER QUEUES

by

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Abstract The approximation of the overflow probability in finite-buffer queues is a practical problem of considerable interest. A new heuristic is given that uses only the equilibrium distribution of the corresponding infinite-buffer queue. Numerical results show an excellent performance of the heuristic.

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1. Result

A practical problem of considerable interest is the calculation of the overflow probability in a finite-capacity queueing system. Consider the GI/G/c/N+c queue with c servers and a finite buffer of capacity N, where any customer finding upon arrival N+c other customers in the system is turned away. It is assumed that the traffic intensity $\rho = \lambda\mu/c$ is less than 1, where λ is the average arrival rate of customers and μ is the average service time of a customer. The assumption $\rho<1$ guarantees that the corresponding infinite-capacity queue GI/G/c has an equilibrium probability distribution $\{\pi_j^{(\infty)}, j=0,1,\ldots\}$ with $\pi_j^{(\infty)}$ denoting the long-run fraction of customers finding upon arrival j other customers present.

A common heuristic for obtaining the overflow probability is

$$P_{app1} = \sum_{j=N+c}^{\infty} \pi_j^{(\infty)}, \tag{1}$$

i.e., the steady-state probability that in the infinite-capacity queue a customer finds upon arrival N+c or more other customers present. A more refined heuristic improving the crude heuristic (1) is

$$P_{app2} = \frac{\pi_{N+c}^{(\infty)}}{1 - \sum_{j>N+c} \pi_j^{(\infty)}}, \tag{2}$$

i.e., the conditional steady-state probability that in the infinite-capacity queue an arrival sees N+c other customers given that no more than N+c customers are present. A quantitative justification of the heuristics (1) and (2) was given in Schweitzer and Konheim [1]. In particular, it was shown that the heuristic (2) is exact for queueing systems with state-dependent Poisson input and exponential services, where customers arrive and depart one at a time.

The purpose of this paper is to present a third heuristic improving the othertwo. The new heuristic is given by

$$P_{app3} = \frac{(1-\rho) \sum_{j=N+c}^{\infty} \pi_j^{(\infty)}}{1-\rho \sum_{j=N+c}^{\infty} \pi_j^{(\infty)}}. \tag{3}$$

What is the rationale behind this heuristic? For the two particular queueing systems M/G/c/N+c with c=1 and M/M/c/N+c, the long-run fraction of time the system is full equals

$$1 - \frac{\sum_{j=0}^{N+c-1} p_j^{(\infty)}}{1 - \rho + \rho \sum_{j=0}^{N+c-1} p_j^{(\infty)}},$$

where $p_j^{(\infty)}$ denotes the time-average probability of having j customers present in the corresponding infinite-capacity queueing system. This result can be found in chapter 4 of Tijms [3] and is based on the fact that for each of the above two queueing models the first N+c-1 state probabilities are proportional to the corresponding state probabilities in the infinite-capacity model. The result above and the fact that Poisson arrivals see time averages motivate the heuristic (3).

Heuristics involving only the equilibrium probabilities for the infinite-capacity model are very useful for practical purposes, since these probabilities have to be computed only once. This is particularly convenient when selecting buffer sizes. Moreover, the customer-average probabilities for the infinite-capacity model are often easier to compute than the corresponding probabilities for the finite capacity model. A further simplification of the heuristics is possible when

$$\pi_j^{(\infty)} \sim \alpha\tau^j \quad \text{for j large enough.} \tag{4}$$

This asymptotic expansion holds true in many queueing systems. Then the decay factor τ can often be computed simply as the root of a nonlinear equation in a single variable. The amplitude factor α is usually not so easy to obtain with the exception of the single-server queue with Poisson arrivals and the multi-server queue with exponential services. However, an approximation or simulation may be used to compute the amplitude factor α.

A numerical investigation of the three heuristics is given in the next section. The numerical results show that the new heuristic (3) improves the other two and performs very well for practical purposes.

2. Numerical results.

A good battlefield for the heuristics (1), (2) and (3) are queueing systems with deterministic arrivals. In table 1 we give the approximate values (1)-(3) and the exact value of the overflow probability for several single-server D/G/1/N+1 queues with Erlang-k services. Similarly, table 2 deals with the multi-server D/M/c/N+c queue. For both queueing systems exact value of the overflow probability was obtained by solving the balance equations of an embedded Markov chain. The infinite capacity versions of these queueing systems are easy to solve. For the D/M/c queue the state probability $\pi_j^{(\infty)}$ equals $\gamma\beta^{j+1-c}$ for $j\geq c-1$, where the constants γ and β are very easy to compute using explicit expressions for the G/M/c queue obtained in Takács [2]. A simple and fast algorithm to compute the state probabilities $\pi_j^{(\infty)}$ for the $D/E_k/1$ queue is given in Tijms and Van de Coevering [4]. This algorithm exploits the asymptotic expansion (4).

Table 1. The overflow probability for the $D/E_k/1/N+1$ queue

		ρ=0.8		ρ=0.9		ρ=0.95	
		N=3	N=8	N=5	N=15	N=10	N=35
E_4	app1	1.87×10^{-3}	1.73×10^{-7}	9.84×10^{-3}	1.84×10^{-6}	1.36×10^{-2}	4.36×10^{-7}
	app2	1.57×10^{-3}	1.46×10^{-7}	5.69×10^{-3}	1.06×10^{-6}	4.65×10^{-3}	1.48×10^{-7}
	app3	3.74×10^{-4}	3.47×10^{-8}	9.92×10^{-4}	1.84×10^{-7}	6.88×10^{-4}	2.18×10^{-8}
	exact	5.49×10^{-4}	5.09×10^{-8}	1.23×10^{-3}	2.28×10^{-7}	7.69×10^{-4}	2.43×10^{-8}
E_2	app1	3.51×10^{-2}	3.38×10^{-4}	9.01×10^{-2}	1.23×10^{-3}	1.11×10^{-1}	6.30×10^{-4}
	app2	2.15×10^{-2}	2.05×10^{-4}	3.34×10^{-2}	4.31×10^{-4}	2.29×10^{-2}	1.18×10^{-4}
	app3	7.22×10^{-3}	6.76×10^{-5}	9.81×10^{-3}	1.23×10^{-4}	6.22×10^{-3}	3.15×10^{-5}
	exact	8.63×10^{-3}	8.05×10^{-5}	1.08×10^{-2}	1.35×10^{-4}	6.54×10^{-3}	3.30×10^{-5}
E_1	app1	1.56×10^{-1}	1.53×10^{-2}	2.76×10^{-1}	3.23×10^{-2}	3.20×10^{-1}	2.41×10^{-2}
	app2	6.43×10^{-2}	5.75×10^{-3}	6.86×10^{-2}	6.40×10^{-3}	4.43×10^{-2}	2.42×10^{-3}
	app3	3.57×10^{-2}	3.10×10^{-3}	3.67×10^{-2}	3.33×10^{-3}	2.30×10^{-2}	1.23×10^{-3}
	exact	3.86×10^{-2}	3.33×10^{-3}	3.84×10^{-2}	3.44×10^{-3}	2.36×10^{-2}	1.26×10^{-3}

The final table 3 gives the approximate values and the exact value of the overflow probability for the M/G/c/N+c queue. Here we substituted in the approximations (1)-(3) the asymptotic expansion (4) for the state probabilities $\pi_j^{(\infty)}$, where the decay factor τ was computed from the characteristic equation for the M/G/c queue and the amplitude factor α was computed from a simple approximation, see pp. 350-351 in Tijms [3]. For small overflow probabilities and nonlight traffic, the use of the asymptotic expansion is justified. It should be noted that the larger the traffic intensity ρ, the earlier the asymptotic expansion (4) applies. The calculations were done for Erlang-2 services (c_s^2=0.5) and hyperexponential services with the normalization of balanced means (c_s^2=2, 4). Here c_s^2 denotes the squared coefficient of variation of the service time.

Table 2. The overflow probability for the D/M/c/N+c queue.

		ρ=0.8		ρ=0.9		ρ=0.95	
		N=5	N=30	N=10	N=70	N=15	N=140
c=5	app1	3.55×10^{-2}	3.24×10^{-7}	7.42×10^{-2}	1.90×10^{-7}	1.71×10^{-1}	4.11×10^{-7}
	app2	1.35×10^{-2}	1.20×10^{-7}	1.52×10^{-2}	3.68×10^{-7}	1.98×10^{-2}	4.05×10^{-8}
	app3	7.31×10^{-3}	6.47×10^{-8}	7.95×10^{-3}	1.90×10^{-8}	1.02×10^{-2}	2.06×10^{-8}
	exact	7.84×10^{-3}	6.93×10^{-8}	8.24×10^{-3}	1.97×10^{-8}	1.04×10^{-2}	2.09×10^{-8}
c=10	app1	2.21×10^{-2}	2.01×10^{-7}	6.11×10^{-2}	1.57×10^{-7}	1.56×10^{-1}	3.76×10^{-7}
	app2	8.31×10^{-3}	7.47×10^{-8}	1.24×10^{-2}	3.03×10^{-8}	1.79×10^{-2}	3.70×10^{-8}
	app3	4.49×10^{-3}	4.03×10^{-8}	6.46×10^{-3}	1.57×10^{-8}	9.16×10^{-3}	1.88×10^{-8}
	exact	4.82×10^{-3}	4.31×10^{-8}	6.70×10^{-3}	1.62×10^{-8}	9.34×10^{-3}	1.91×10^{-8}
c=25	app1	7.35×10^{-3}	6.70×10^{-8}	4.03×10^{-2}	1.03×10^{-7}	1.30×10^{-1}	3.13×10^{-7}
	app2	2.74×10^{-3}	2.49×10^{-8}	8.03×10^{-3}	1.99×10^{-8}	1.45×10^{-2}	3.08×10^{-8}
	app3	1.48×10^{-3}	1.34×10^{-8}	4.18×10^{-3}	1.03×10^{-8}	7.42×10^{-3}	1.57×10^{-8}
	exact	1.59×10^{-3}	1.44×10^{-8}	4.33×10^{-3}	1.07×10^{-8}	7.56×10^{-3}	1.59×10^{-8}

Table 3 The overflow probability for the M/G/c/N+c queue with c=10.

		ρ=0.8		ρ=0.9		ρ=0.95	
		N=10	N=25	N=20	N=50	N=25	N=100
c_s^2=0.5	app1	2.37×10^{-2}	2.89×10^{-4}	4.26×10^{-2}	6.45×10^{-4}	1.53×10^{-1}	9.18×10^{-4}
	app2	6.15×10^{-3}	7.35×10^{-5}	5.77×10^{-3}	8.42×10^{-5}	1.18×10^{-2}	6.06×10^{-5}
	app3	4.84×10^{-3}	5.78×10^{-5}	4.43×10^{-3}	6.46×10^{-5}	8.94×10^{-3}	4.60×10^{-5}
	exact	5.12×10^{-3}	6.11×10^{-5}	4.55×10^{-3}	6.62×10^{-5}	9.08×10^{-3}	4.67×10^{-5}
c_s^2=2	app1	8.20×10^{-2}	9.95×10^{-3}	1.56×10^{-1}	1.96×10^{-2}	3.42×10^{-1}	2.69×10^{-2}
	app2	1.18×10^{-2}	1.28×10^{-3}	1.22×10^{-2}	1.34×10^{-3}	1.70×10^{-2}	9.19×10^{-4}
	app3	1.75×10^{-2}	1.90×10^{-3}	1.81×10^{-2}	2.00×10^{-3}	2.53×10^{-2}	1.38×10^{-3}
	exact	1.46×10^{-2}	1.51×10^{-3}	1.63×10^{-2}	1.82×10^{-3}	2.37×10^{-2}	1.32×10^{-3}
c_s^2=4	app1	1.33×10^{-1}	3.73×10^{-2}	2.60×10^{-1}	7.59×10^{-2}	5.56×10^{-1}	1.22×10^{-1}
	app2	1.24×10^{-2}	3.14×10^{-3}	1.39×10^{-2}	3.29×10^{-3}	2.44×10^{-2}	2.77×10^{-3}
	app3	2.98×10^{-2}	7.69×10^{-3}	3.39×10^{-2}	8.15×10^{-3}	5.89×10^{-2}	6.90×10^{-3}
	exact	2.14×10^{-2}	5.10×10^{-3}	2.72×10^{-2}	6.70×10^{-3}	3.64×10^{-2}	5.18×10^{-3}

Summarizing, the numerical results indicate that the new heuristic (3) performs very well for practical purposes. Also, note that the crude heuristic (1) commonly used in practice may perform very poorly. In all cases the value of the new heuristic is of the same order of magnitude as the exact value of the overflow probability. This is exactly what is needed when a heuristic is used for dimensioning the buffer size so that the overflow probability does not exceed a prespecified value.

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