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Gerard van der Laan

Pieter H.M. Ruys

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A M S T E R D A M

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Gerard van der Laan[1]

Pieter H.M. Ruys[2]

Dolf J.J. Talman[2]

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1) Department of Economics and Econometrics, Free University, De Boelelaan 1105, 1081 HV Amsterdam, The Netherlands.

2) Department of Econometrics, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands.

Signaling devices for the supply of semi-public goods

Abstract.

This paper deals with the concept of a semi-public good. These semi-public goods are characterized by the fact that their use is being supplemented by specific private goods. The consumption of this complementary private good is constrained by an individual quantity constraint for each individual agent. The quantity constraints depend on the level of the semi-public good. For instance, car driving is limited by the level of the road system. This approach allows us to design economic institutions which carry out price discrimination among users of a semi-public good. People who are seriously hampered by too small a provision of a public good, because it constrains their use of the private commodity, are willing to pay a mark-up on the price for the latter one if this mark-up is spent for expanding the provision of the public good. In the model the availability of a public good is planned and organized by a central planner. The consumer's willingness to pay an individual mark-up on the price of a private commodity reflects his preferences for the availibility of the public good. These mark-ups are collected by the private goods industry and transferred to the central planner in order to cover the costs of the public good infrastructure. This framework of a private industry and a central planner providing semi-public goods is called an industrial economy. The main issue of this paper is to prove the existence of an equilibrium in such an economy

1. Introduction.

The fundamental problem associated with pure public goods is the revelation of preferences. In real life this problem is ususally solved by political mechanisms, economic mechanisms being too cumbersome or impractical. Only the financing of public goods remained an economic problem. The theory of public finance has therefore been preoccupied with extracting money from the market economy to supply the government with sufficient funds.

In this paper we propose a different approach. it is our purpose to bring the optimal provision and financing of a catagory of public goods into the domain of economics. We have succeeded only for a special type of public goods, which we call semi-public goods. These semi-public goods are characterized by the fact that their use is being supplemented by specific private goods. Examples are: roads and cars; pollution and polluters; infrastructure and users; hospitals and medical services by doctors; quality and products; information and information carriers. In short, we relate infrastructural commodities with final goods that are based on this infrastructure.

The idea is that people may be constrained in their individual use of a supplementary good, caused by a shortage of the related semi-public good. People do not need to realize that the cause of the shortage is to be found in the semi-public good: they are prepared to pay higher prices for the supplementary private commodity. This willingness-to-pay is seen as a signal given by the consumers indicating the scarcity of the underlying semi-public good.

We assume that the industry is able to observe these signals and to discern groups of consumers giving different signals. Some people are more eager to be served than other people, are protesting louder and are willing to py a higher price for the private good if it is being served to them. These supplementary goods are signalling devices for the supply of semi-public goods. If this is the case, we show that:

- the social benefits of the semi-public good for the use of some supplementary private goods can be determined.
- efficient prices can be derived, which allow for decentralization of consumption and production decisions to a market.
- the semi-public good is approximately budget-neutrally financed by the market.
- the optimal supply of the semi-public good can be computed.

This paper is a contribution to the microeconomic public regulation of behavior, substituting political or juridicial measures into economic devices. It enables us to organize an economy with local public goods consistent with a global market.

The main field of application will be the organization of an industry. In the sequel we will use public transportation as an example, although other complex goods such as a clean environment, health care or education might also be taken. The theory is relevant for both non-profit institutions and profit maximizing firms that are faced with the problem of finding performance criteria, which are faced with a shift from input-financing to output-financing. In order to judge both the effectiveness and the efficiency of production, they are forced to develop output criteria. These may be complementary private goods.

We use the concept of a semi-public good, introduced by **Ruys** [5], see also **Ruys and van der Laan** (6) with a slight adaption of the formal definition. Here, a semi-public good is defined as a public good having a complementary private commodity. For each agent i, the amount y^i of his consumption of the private commodity and the amount z of availability of the public good are related to each other by an individual inequality constraint $y^i \leq \bar{y}^i(z)$ for each agent i. This constraint might be implicitly expressed in the consumer's utility function or the producer's production function. But the explicit formulation makes it possible to distinguish between whether an individual constraint is binding or not. If for some agent, say consumer i, the constraint is binding, then an increase of z has a direct effect on his demand because of the fact that z appears in the consumer's utility function, but also it has an indirect effect through the weakening of the constraint. The price for raising z offered by a truth-telling consumer will reflect the impact of both effects on his utility. The part reflecting the constraint will show up as a mark-up on the market price the consumer is willing to pay for the private commodity. If no agent in the economy feels himself constrained in the complementary commodity, the semi-public good reduces to a pure public public good with, if desired, Lindahl prices. In general, the definition of a semi-public good is relevant only if the constraints are binding for a considerable number of agents.

The main advantage of this approach is that economic institutions can be designed which make price discrimination possible among users of a semi-public good. People who are seriously hampered by too small a provision of a public good, because it constrains their use of the private commodity, are thought of forming (political) pressure groups to expand its provision, or are informing the industry otherwise. They are also willing to pay a mark-up on the price of the private commodity. In this way the private commodity is a signalling device for the semi-public good.

We will explore a model in which there is just one industry producing the complementary private commodity to a public good. The infrastructure of this public good is planned and organized by a central planner. The consumer's willingness to pay an individual mark-up on the price of the private commodity reflects his preferences for the infrastructure of the public good. This mark-up is collected by the private goods industry and transferred to the central planner in order to cover the costs of the public good infrastructure. As an alternative the private goods industry may levy a uniform mark-up on the prices of the private commodities to provide an infrastructure necessary for using their product. We call this framework of a central planner and private firms providing the complementary private good an industrial economy.

It is evident that there are many spill-over effects resulting from any decision about the provision of a semi-public good. This calls for a general equilibrium approach, with an associated fixed point or zero point formulation. This paper is organized as follows. In section 2 the mathematical model is given. In that section we also state the first order conditions for a Pareto efficient allocation. The institutional framework to reach a Pareto efficient allocation is given in section 3. In section 4 we give the existence proof of an equilibrium in an industrial economy. Finally, in section 5 we make some remarks about the implementation and computation of an equilibium.

2. The mathematical model.

We consider a model of an industrial economy with one semi-public goods and one complementary private good. For example, the public good is a road system that is used by private cars, or a railway infrastructure that is used by train passengers. There is one other (composite) private commodity. A (possibly private) producer plans and organizes the level of the infrastructure of the semi-public good, taking into account the wishes of the (transportation) industry.

Let there be h consumers, indexed by $i=1,...,h$. Each consumer i has a utility function $u^i(x^i,y^i,z)$ on $X^i=R^3_+$, where x^i is the consumption of the composite private commodity, y^i the consumption of the complementary private good and z the amount of the semi-public good. Consumer i has an initial endowment w^i of the composite private good. Furthermore, each consumer i faces an individual quantity constraint on the consumption of the complementary private good. That means, for instance, each consumer is constrained in his' or her's car driving because of the limitations of the road system. So, we assume that there exist nonnegative increasing constraint functions $\bar{y}^i$ for i=1,...,h, such that given z the consumption y^i of consumer i is restricted by

$$y^i \leq \bar{y}^i(z). \tag{2.1}$$

The industry is aware of these (subjective) constraints because it observes rationing in the demand functions. We assume that there is only one firm producing the complementary private good. The production function of the firm is given by

$$F^a(x^a,y^a) \leq 0, \tag{2.2}$$

with $x^a \leq 0$ the input of the composite private good and $y^a \geq 0$ the output of the complementary commodity.

The enterprise producing the (public) infrastructure is characterized by the production function

$$F^b(x^b,z^b) \leq 0, \tag{2.3}$$

with $x^b \leq 0$ the input of the composite private good and z^b the output of the semi-public good. There is an initial level $\bar{z}$ of the semi-public good. So, after production, the total level is $z = z^b + \bar{z}$ with $z^b \geq 0$. We define for all i, $y^i(z^b) = \bar{y}^i(\bar{z}+z^b) = \bar{y}^i(z)$.

We assume that this economy, denoted by $E = \{(u^i,y^i,)\ i=1,...,h,\ F^a,\ F^b,\ w,\ \bar{z}\}$ is *regular*, i.e., the utility and production functions and the constraint functions are continuously differentiable, the utility functions u^i are monotonically increasing and strictly quasi-concave, the productions functions are strictly concave and satisfy $F^a(0,0) = 0$ and $F^b(0,0) = 0$, and $w = (w^1,...,w^h)$ and $\bar{z}$ are strictly positive.
We are now ready to give some definitions.

Definition 2.1. An allocation $e = \{(x^i,y^i),\ i=1,...,h,\ x^a,\ y^a,\ x^b,\ z^b\}$ is in the set A of *feasible* allocations if the constraints (2.1)-(2.3) hold with in (2.1) $z = \bar{z} + z^b$, and if

$$\Sigma_i\ x^i \leq \Sigma_i\ w^i + x^a + x^b \tag{2.4}$$

$$\Sigma_i\ y^i \leq y^a. \tag{2.5}$$

Observe that this definition includes the subjective constraints (2.1) $y^i \leq \bar{y}^i(z) = y^i(z^b)$. The quantity constraints (2.4) and (2.5) state that total demand is less than or equal to total supply.

Definition 2.2. A feasible allocation e is *efficient* if there is a social welfare function $W(u^1,...,u^h)$, strictly increasing in u^i, $i = 1,...,h$, such that e maximizes the social welfare over the set A of feasible allocations.

According to Definition 2.2 the necessary conditions for an allocation of a regular economy to be efficient follow from the maximization problem,

$$\max W(u^i(x^i,y^i,z+z^b),\ i = 1,...,h), \tag{2.6}$$

such that, with the shadow prices of the constraints between brackets,

(α^i) $\quad y^i - y^i(z^b) \le 0 \quad i=1,...,h$

(λ^a) $\quad F^a(x^a,y^a) \le 0$

(λ^b) $\quad F^b(x^b,z^b) \le 0$

(μ^1) $\quad \Sigma_i\, x^i \le \Sigma_i\, w^i + x^a + x^b$

(μ^2) $\quad \Sigma_i\, y^i \le y^a$

(μ^3) $\quad z^b \ge 0.$

Differentiating the corresponding Lagrange function gives with respect to the variable between brackets:

(x^i) $\quad (\delta W/\delta u^i)(\delta u^i/\delta x^i) - \mu^1 = 0, \quad i = 1,...,h \quad (2.7)$

(y^i) $\quad (\delta W/\delta u^i)(\delta u^i/\delta y^i) - \alpha^i - \mu^2 = 0, \quad i = 1,...,h \quad (2.8)$

(x^a) $\quad -\lambda^a\, \delta F^a/\delta x^a + \mu^1 = 0 \quad (2.9)$

(y^a) $\quad -\lambda^a\, \delta F^a/\delta y^a + \mu^2 = 0 \quad (2.10)$

(x^b) $\quad -\lambda^b\, \delta F^b/\delta x^b + \mu^1 = 0 \quad (2.11)$

(z^b) $\quad \Sigma_i\, (\delta W/\delta u^i)(\delta u^i/\delta z^b) + \Sigma_i\, \alpha^i\, \delta y^i/\delta z^b$

$$-\lambda^b\, \delta F^b/\delta z^b + \mu^3 = 0. \tag{2.12}$$

with all shadow prices nonnegative. Let us consider now an efficient allocation. For simplicity we assume that all constraints hold with equality except possibly for some of the individual constraints $y^i - y^i(z^b) \le 0$, $i=1,...,h$, on the consumption of the complementary private good, or for the positive production constraint $z^b \ge 0$. If $y^i - y^i(z^b) < 0$ then $\alpha^i = 0$. Hence with the composite private commodity taken as the numeraire, we obtain from (2.7)-(2.10) that in this case the first order condition for an efficient consumption of the complementary private good of i becomes,

$$\frac{\delta u^i/\delta y^i}{\delta u^i/\delta x^i} = \frac{\delta F^a/\delta y^a}{\delta F^a/\delta x^a} \qquad \text{if } y^i - y^i(z^b) < 0.$$

This equation implies that for consumer i this commodity is then a private good, i.e., the MRS is equal to the MRT between this good and the numeraire commodity. If $\alpha^i = 0$ for all i, then no consumer feels himself constrained in the use of complementary private good and we obtain from (2.7), (2.11) and (2.12) that the first order condition for the public good becomes

$$\Sigma_i \frac{\delta u^i/\delta z^b}{\delta u^i/\delta x^i} = \frac{\delta F^b/\delta z^b}{\delta F^b/\delta x^b} \qquad \text{if } y^i - y^i(z^b) < 0 \text{ for all i.}$$

However, if $\alpha^i > 0$ for some i, then consumer i is willing to pay a mark-up on the MRT of the complementary commodity in order to subsidize an expansion of the infrastructure. This is reflected in the derivitive of the Lagrange function with respect to y^i through the Lagrange multiplier α^i. So, the first order conditions for the consumption of the complementary private good become,

$$\frac{\delta u^i/\delta y^i}{\delta u^i/\delta x^i} \geq \frac{\delta F^a/\delta y^a}{\delta F^a/\delta x^a} \qquad \perp\ y^i - y^i(z^b) \leq 0,\ i = 1,\ldots,h, \tag{2.13}$$

where $a \geq b \perp c \leq d$ means: $a \geq b$, $c \leq d$ and $(a-b)(c-d) = 0$.

The mark-ups for expanding the public good appear in the first order conditions for the public good. From (2.7)-(2.10) it follows that

$$\frac{\alpha^i}{\mu^1} = \frac{\delta u^i/\delta y^i}{\delta u^i/\delta x^i} - \frac{\delta F^a/\delta y^a}{\delta F^a/\delta x^a}$$

so that the first order condition for the production of the public goods follows from (2.7), (2.11) and (2.12) by substituting α^i/μ^1 and becomes

$$\Sigma_i \frac{\delta u^i/\delta z^b}{\delta u^i/\delta x^i} + \Sigma_i \left[\frac{\delta u^i/\delta y^i}{\delta u^i/\delta x^i} - \frac{\delta F^a/\delta y^a}{\delta F^a/\delta x^a}\right] \delta y^i/\delta z^b = \frac{\delta F^b/\delta z^b}{\delta F^b/\delta x^b} - \frac{\mu_3}{\mu_1}. \tag{2.14}$$

We see that the sum of the MRS plus the sum of the mark-ups of the consumers is equal to the MRT of the public good minus μ_3/μ_1. If all mark-ups are equal to zero, then the public good behaves as a pure public good. We have that $\mu_3 > 0$ if $z^b = 0$ and $\mu_3 = 0$ if $z^b > 0$, so that (2.14) becomes

$$z^b \geq 0 \perp \Sigma_i \frac{\delta u^i/\delta z^b}{\delta u^i/\delta x^i} + \Sigma_i \frac{\delta u^i/\delta y^i}{\delta u^i/\delta x^i} - \frac{\delta F^a/\delta y}{\delta F^a/\delta x^a} \delta y^i/\delta z^b \leq \frac{\delta F^b/\delta z^b}{\delta F^b/\delta x^b} \qquad (2.15)$$

So, if the production is positive then the sum of the MRS plus the sum of the mark-ups of the consumers is equal to the MRT of the public good.

The main advantage of introducing semi-public goods in this way is that an industrial economy can discriminate between agents who are and who are not constrained by the infrastructure, because it can observe demand-behavior. This information can solve partially (and sometimes completely) the difficult problem of determining the individual contributions to the provision of a public good.

3. The institutional framework.

In this section we describe the institutional framework under which an industrial equilibrium can be formulated satisfying the first order conditions for efficiency. This institutional framework is the private ownership industrial economy. In the economy *E* there are two private good markets in operation: one for both the composite private good and the complementary private good. The demands and supplies on these markets depend on their prices, p_x and p_y respectively. We take the private commodity as the numeraire commodity and set its price p_x equal to one. In an efficient allocation $p_y = (\delta F^a/\delta y^a)/(\delta F^a/\delta x^a)$, i.e., p_y equals the MRT. For the third commodity, the semi-public good, the situation is more complicated.

We assume that the industry is able to discriminate among consumers who are constrained in the use of the complementary private good and who are not. At some efficient allocation e, let, for i=1,...,h,

$$t^i(e) = \frac{\delta u^i/\delta y^i}{\delta u^i/\delta x^i} - \frac{\delta F^a/\delta y^a}{\delta F^a/\delta x^a} = (\delta u^i/\delta y^i)/(\delta u^i/\delta x^i) - p_y \qquad (3.1)$$

be the willingness of consumer i to pay for the weakening of the constraint $y^i(z^b)$. Then $t^i(e)\delta y^i/\delta z^b$ is his willingness to pay for the expansion of the infrastructure. If $y^i < y^i(z^b)$ then the MRS of consumer i equals the MRT and hence $t^i(e) = 0$. Next, at some allocation e, let us denote the marginal rate of substitution of consumer i between the semi-public good and the composite good by $p^i(e)$, i=1,...,h. The planner's task of finding the desired level of the infrastructure is to make available an amount z such that the sum of these MRS's plus the total willingness to pay (marginal social benefits of the public good) is equal to the marginal rate of transformation (marginal social cost), $p_z(e)$.

Planner's problem: Find z such that

$$\Sigma_i\, [p^i(e) + t^i(e)\delta y^i/\delta z^b] \leq p_z(e) \perp z^b \geq 0. \tag{3.2}$$

The price to be paid by the planner for each unit of the production of the public good, p_z, equals the MRT. On the other hand, the revenues of the planner consist of the consumers' contributions p^i per unit of the produced public good, and the mark-up t^i per unit of complementary private good. If the revenues are not enough to cover the marginal cost price for any $z^b > 0$, then z^b must be taken equal to zero. The planner's profit π^0 is therefore equal to

$$\pi^0 = \Sigma_i\, p^i z^b + \Sigma_i\, t^i y^i - p_z z^b. \tag{3.3}$$

Using (3.2) and with $p^i(e) = p^i$, $t^i(e) = t^i$ and $p_z(e) = p_z$, π^0 becomes

$$\pi^0 = \Sigma_i\, t^i y^i - t^i z^b \delta y^i/\delta z^b = \Sigma_i\, t^i y^i (1 - \epsilon^i(z^b)) \tag{3.4}$$

with $\epsilon^i(z^b) = (\delta y^i/\delta z^b) z^b/y^i$ the consumer's i individual infrastructure elasticity of the demand for the complementary private good at the level z. So, the planner's profit equals zero if for all i, $\epsilon^i = 1$ or $t^i = 0$. If for all i, $y^i(0) = 0$ and y^i is a convex function, then $\epsilon^i \geq 1$ for all i and π^0 is nonpositive. On the other hand π^0 is nonnegative if for all i, $y^i(0) \geq 0$ and y^i is a concave function and hence $\epsilon^i \leq 1$ for all i. If for all i, $y^i(z^b) = a^i z^b$ for some positive a^i, then $\pi^0 = 0$.

To complete the description of the economy, we assume that the private firms are profit maximizing producers. We denote the respective profits by $\pi^a(p_y)$ and $\pi^b(p_z)$. All profits are distributed among the consumers including the planner's profit, with, for i=1,...,h and $r \in \{0,a,b,\}$, ϕ^{ir} the share of consumer i in the profit of firm (or planner) r. All shares are nonnegative and $\Sigma_i\, \phi^{ir}=1$ for all r. Recall that the price p_x of the numeraire composite commodity is equal to one. So, at prices p_y and p_z, and given the planner's profit π^0, the budget of consumer i is given by

$$B^i(p,\pi^0) = w^i + \phi^{i0}\pi^0 + \phi^{ia}\pi^a(p_y) + \phi^{ib}\pi^b(p_z).$$

We are now able to define an industrial equilibrium for the economy E.

Definition 3.1. An industrial equilibrium for the economy E is an allocation $e = \{(x^i, y^i), i=1,...,h, x^a, y^a, x^b, z^b\}$, prices $p_y \geq 0$ and $p_z \geq 0$ for the complementary private commodity and the public good, individual public good prices $p^i \geq 0$ and mark-ups $t^i \geq 0$, i=1,...,h, and a planner's profit π^0, such that

1) for all i, (x^i,y^i,z^b) maximizes $u^i(\hat{x}^i,\hat{y}^i,z+\hat{z}^b)$ under the budget constraint

$$\hat{x}^i + (p_y+t^i)\hat{y}^i + p^i\hat{z}^b \leq B^i(p,\pi^0);$$

2) each producer maximizes profit subject to his technical constraint, i.e.,

$\pi^a(p_y) = x^a + p_y y$ is the maximum of $\hat{x}^a + p_y \hat{y}^a$ such that $F^a(\hat{x}^a,\hat{y}^a) \leq 0$,
$\pi^b(p_z) = x^b + p_z z^b$ is the maximum of $\hat{x}^b + p_z \hat{z}^b$ such that $F^b(\hat{x}^b,\hat{z}^b) \leq 0$;

3) for all i: $t^i \geq 0 \perp y^i \leq y^i(z^b)$;

4) $\Sigma_i x^i = \Sigma_i w^i + x^a + x^b$

$\Sigma_i y^i = y^a$;

5) $z^b \geq 0 \perp \Sigma_i [p^i + t^i \delta y^i(z^b)/\delta z^b] \leq p_z$, i.e., the planner's constraint is satisfied,

and

$\pi^0 = \Sigma_i t^i y^i(1 - \epsilon^i(z^b))$.

Observe that the planner's profit appears in the budget of the consumer and depends on the values of the consumer's decision. Therefore it is taken explicitly in the definition of an equilibrium. In an equilibrium the availability of the public good is completely determined by the planner. So, actually the consumers do not maximize their utility over z. Instead, the prices p^i are determined such that for all i, z^b is optimal under p^i. The same reasoning holds for the public good's producer, who determines p_z given the amount z^b. The third condition has analogies in fixed price theory, from which it is well-known that quantity-constrained allocations can be sustained by virtual prices (see e.g. **Neary** and **Roberts** [4]). Here condition 3) says that a consumer is not willing to pay a mark-up on the cost price of a commodity if he or she is not constrained in the use of that commodity.

Proposition 3.2. An industrial equilibrium allocation satisfies the first order conditions for Pareto efficiency.

Proof. First, from 3), 4) and the technical constraints in the profit maximization conditions it follows that the allocation satisfies (2.1) - (2.5) and hence the allocation is feasible. Secondly, it follows from the utility maximization in 1), the maximization of π^a in 2) and the complementarity condition 3) that condition (2.13) of Pareto optimality is satisfied, while condition (2.15) follows from the utility maximization, the maximization of π^a and π^b, and from the planner condition 5).

So, taking the second order conditions for granted we have that an industrial equilibrium allocation is Pareto optimal.

4. Existence of an industrial equilibrium.

In this section we give a formal proof of the existence of an industrial equilibrium. This proof is not based on the implementation of the institutional framework of an industrial economy. We will address the latter problem in the next section.

To prove the existence, we define positive integers A, B and C such that $A > \Sigma_i w^i$, $B > \max\{y|F^a(x^a,y^a) \leq 0$ and $-x^a \leq A\}$ and $C > \bar{z} + \max\{z|F^b(x^b,z^b) \leq 0$ and $-x^b \leq A\}$. So, A is greater than the total initial endowment and B and C exceed the maximal production of the private good and the public good.

We assume that the economy E is regular. Furthermore, we make the following assumptions.

Assumption 4.1. $(\delta F^a/\delta y^a)/(\delta F^a/\delta x^a)$ is bounded for all finite x^a and $(\delta F^b/\delta z^b)/(\delta F^b/\delta x^b)$ is bounded for all finite x^b.

Assumption 4.2. For all i, the constraint function y^i is continuously differentiable, $\epsilon^i(0) = \lim_{z^b \to 0} (\delta y^i/\delta z^b)z^b/y^i$ exists, and $\epsilon^i(z^b) = (\delta y^i/\delta z^b)z^b/y^i(z^b)$ is at least one at any $z^b \geq 0$.

Assumption 4.3. For all i:

1) $y^i(\bar{z}) > 0$,
2) $(\delta u^i/\delta y^i)/(\delta u^i/\delta x^i)$ is decreasing in y^i and increasing in x^i,
3) $(\delta u^i/\delta z)/(\delta u^i/\delta x^i)$ is decreasing in z and increasing in x^i.

Asumption 4.1 only says that the marginal rate of transformation is bounded. This guarantees that the input goes to infinity if the price ratio goes to infinity. Assumption 4.2 on the constraint function guarantees that the planner's profit is nonnegative at any solution z^b of the level of the infrastructure. The nonnegativity of this profit guarantees positive incomes for the individuals, which is needed to prove existence. However, profits may even be negative, as long as the planner's deficit is small compared to the initial endowments and hence the individual incomes remain positive. The first part of Assumption 4.3 is a technical assumption saying that the current level $\bar{z}$ allows for a positive use. The other parts say the marginal utility is decreasing.

Suppose that $((x^i, y^i), i=1,...,h, x^a, y^a, x^b, z^b)$ is an industrial equilibrium allocation with $p_x = 1$, p_y, p_z, p^i, $i = 1,...,h$ and t^i, $i=1,...,h$, the corresponding prices and mark-ups, and with π^0 the corresponding planner's profit. The budget restriction of consumer i is homogenous of degree zero in the prices, the mark-up t^i and the profits. Moreover the producers' profits are homogeneous of degree one in the prices and the producers' decisions homogeneous of degree zero. Finally, both equations in equilibrium condition 5) are homogeneous of degree one in prices, mark-ups and the planner's profit. Hence if the allocation $\{(x^i, y^i), i=1,...,h, x^a, y^a, x^b, z^b\}$ is an industrial equilibrium allocation supported by $p_x = 1$, p_y, p_z, p^i and t^i, $i=1,...,h$, and π^0, then it is also supported by $p_x = \lambda$, λp_y, λp_z, λp^i and λt^i, $i=1,...,h$, and $\lambda\pi^0$ for any $\lambda > 0$. So, we can restrict the vector of prices $p = (p_x, p_y, p_z)$ in the nonnegative orthant of the 3-dimensional Euclidean space to the set of vectors in the 2-dimensional unit simplex

$$S^2 = \{p \in R^3_+ \mid \Sigma_k p_k = 1\}.$$

Now, let $q \in R^{2h+3}$ be a vector $(p_x, p_y, p_z, p^1,...,p^h, t^1,...,t^h)^T$ of prices and mark-ups with with $(p_x, p_y, p_z) \in S^2$, i.e., $q \in S^2 \times R^{2h}$. Let $(x^a(q), y^a(q))$ be the solution to

$$\text{maximize } p_x x^a + p_y y^a \text{ such that } F^a(x^a,y^a) \le 0 \text{ and } -x^a \le A,$$

and let $\pi^a(q)$ be the corresponding profit $p_x x^a(q) + p_y y^a(q)$.
Analogously, let $(x^b(q), z^b(q))$ be the solution to

$$\text{maximize } p_x x^b + p_z z^b \text{ such that } F^b(x^b,z^b) \le 0 \text{ and } -x^b \le A,$$

and let $\pi^b(q)$ be the corresponding profit $p_x x^b(q) + p_z z^b(q)$.
Furthermore, let $\pi^0(q) = \Sigma_i\ t^i y^i(z^b(q))(1 - \epsilon^i(q))$, with $\epsilon^i(q) = (\delta y^i(z^b(q))/\delta z^b) z^b(q)/y^i(z(q)))$. Under Assumption 4.2, $\pi^0(q)$ is nonnegative and under the regularity assumption also the profits $\pi^a(q)$ and $\pi^b(q)$ are nonnegative. Moreover, under the regularity assumption the profit maximization problems have unique solutions being continuous in q if $p_x > 0$, and hence all profits are continuous in q if $p_x > 0$. Now, let $(x^i(q), y^i(q), z^i(q))$ be the solution to

$$\text{maximize } u^i(x^i,y^i,z^i) \text{ under the budget constraint}$$
$$p_x x^i + (p_y+t^i)y^i + p^i(z^i-z) \le B^i(q),$$

with $B^i(q) = p_x w^i + \phi^{i0}\pi^0(q) + \phi^{ia}\pi^a(q) + \phi^{ib}\pi^b(q)$,
and under the constraints $x^i \le A$, $y^i \le B$ and $z^i \le C$. Since $w^i > 0$ and all profits are nonnegative, consumer i has a positive income for all q with $p_x > 0$. Since all profits are continuous in q if $p_x > 0$, $B^i(q)$ is a continuous correspondence at all q with $p_x > 0$ (**Debreu** [1]. From Assumption 4.1 it follows that there exists an L^a such that $x^a(q) = -A$ for $p_y/p_x \ge L^a$. Analogously, there exists an L^b such that $x^b(q) = -A$ for $p_z/p_x \ge L^b$. Since $q \in S^2 \times R^{2h}$ and hence $p_x + p_y + p_z = 1$, there exists an $\nu > 0$, such that $p_y/p_x \ge L^a$ or $p_z/p_x \ge L^b$ if $p_x \le \nu$. So, the demand for the composite private good exceeds the total initial endowment for $p_x \le \nu$. Therefore we restrict the vectors q to the set $S^2_\nu \times R^{2h}$ with $S^2_\nu = \{p \in R^3_+ \mid p_x + p_y + p_z = 1 \text{ and } p_x \ge \nu\}$. Then $B^i(q)$ is a continuous correspondence at all $q \in S^2_\nu \times R^{2h}$ and hence

under the regularity assumption the utility maximization problem has a unique solution ($x^i(q)$, $y^i(q)$, $z^i(q)$) being continuous in q. Moreover the budget constraint is satisfied with equality because of the monotonicity of the utility functions if at least one of the quantity constraints $x^i \leq A$, $y^i \leq B$ and $z^i \leq C$ is not binding. Observe that $y^i = B$ if $p_y+t^i = 0$, and $z^i = C$ if $p^i = 0$.

Next we define:

$$f_x(q) = \Sigma_i x^i(q) - \Sigma_i w^i - x^a(q) - x^b(q),$$
$$f_y(q) = \Sigma_i y^i(q) - y^a(q)$$
$$f_z(q) = \Sigma_i \{p^i + t^i \delta y^i(z^b(q))/\delta z^b] - p_z,$$
$$f^i(q) = z^i(q) - z^b(q) - z, \qquad i = 1,\ldots,h,$$
$$\tau^i(q) = y^i(q) - y^i(z^b(q)), \qquad i = 1,\ldots,h.$$

We now define the function ξ: $S^2_\nu \times R^{2h} \rightarrow R^{2h+3}$, by $\xi(q) = [f_x(q), f_y(q), f_z(q), f^1(q),\ldots,f^h(q), \tau^1(q),\ldots,\tau^h(q)]^T$. We will prove that a point q^* such that $\xi(q^*) \leq 0$ yields an equilibrium and that such a point q^* exists.

From Assumption 4.3 it follows that there exists positive numbers M^i and T^i such that $z^i(q) < z$ if $p^i \geq M^i$ and $y^i(q) \leq y^i(z)$ if $t^i \geq T^i$, $i = 1,\ldots,h$. We therefore restrict the domain of the function ξ to the compact set $S^2_\nu \times R^{2h}(M,T)$, with $R^{2h}(M,T) = \{p^1,\ldots,p^h, t^1,\ldots,t^h \mid 0 \leq p^i \leq M^i, 0 \leq t^i \leq T^i, i = 1,\ldots,h\}$. We now apply the following well-known lemma.

<u>Lemma 4.4.</u> Let f be a continuous function from a compact set S in R^k to R^k. Then there exists a point $q^* \in R^k$, such that $q^T f(q^*) \leq q^{*T} f(q^*)$ for all $q \in S$.

We call q^* a stationary point of f on S.

<u>Lemma 4.5.</u> Let q^* be a stationary point of ξ on $S^2_\nu \times R^{2h}(M,T)$. Then $\xi(q^*) \leq 0$.

<u>Proof.</u> Let

$$L(q) = p_x f_x(q) + p_y f_y(q) + z^b(q) f_z(q) + \Sigma_i p^i f^i(q) + \Sigma_i t^i \tau^i(q).$$

From the definition of $f_x(q)$, $f_y(q)$, $z(q)$, $f_z(q)$, $f^i(q)$ and $\tau^i(q)$, $i = 1,\ldots,h$, $\pi^a(q)$, $\pi^b(q)$, $\pi^0(q)$ and the budget constraints it follows that $L(q) \leq 0$ for all q and $L(q) = 0$ if for all i the budget constraints are satisfied with equality. Now, let $q^* = (p_x^*, p_y^*, p_z^*, p^{1*},\ldots,p^{h*}, t^{1*},\ldots,t^{h*},)^T$ be a stationary point. From $q^T f(q^*) \leq q^{*T} f(q^*)$ for all $q \in S^2_\nu \times R^{2h}(M,T)$ it follows that $f^i(q^*) \leq 0$ if $p^{i*} = 0$, $f^i(q^*) = 0$ if $0 \leq p^{i*} \leq M^i$ and $f^i(q^*) \geq 0$ if $p^{i*} = M^i$. If $p^i = M^i$ then $z^i(q) < z$ and hence $f^i(q) < 0$, so that at a stationary point $p^{i*} \geq 0 \perp f^i(q^*) \leq 0$ for all i. Analogously we have that $t^{i*} \geq 0 \perp \tau^i(q^*) \leq 0$ for all i. From this it also follows that $p^{i*}f^i(q^*) = 0$ and $t^{i*}\tau^i(q^*) = 0$ for all i and hence

$$p_x^*f_x(q^*) + p_y^*f_y(q^*) + z^b(q^*)f_z(q^*) \leq 0 \tag{4.1}$$

while $p_x^* + p_y^* + p_z^* = 1$. Moreover the stationary point definition implies that for all $(p_x, p_y, p_z,)^T \in S^2_\nu$,

$$p_xf_x(q^*) + p_yf_y(q^*) + p_zf_z(q^*) \leq p_x^*f_x(q^*) + p_y^*f_y(q^*) + p_z^*f_z(q^*) \tag{4.2}$$

From this we obtain that also $f_x(q^*) \leq 0$, $f_y(q^*) \leq 0$ and $f_z(q^*) \leq 0$ as follows. From (4.2) it follows that $f_x(q^*) = f_y(q^*) = f_z(q^*)$ if $(p_x^*, p_y^*, p_z^*)^T \in \text{int } S^2_\nu$, i.e, if $p_x^* > \nu$, $p_y^* > 0$ and $p_z^* > 0$. Hence, it follows with (4.1) that $f_x(q^*) = f_y(q^*) = f_z(q^*) \leq 0$ if $(p_x^*, p_y^*, p_z^*)^T \in \text{int } S^2_\nu$.

Now, suppose $p_x^* = \nu$. Then (4.2) implies that $f_x(q^*) \leq \min\{f_y(q^*), f_z(q^*)\}$ if both $p_y^* > 0$ and $p_z^* > 0$, $f_x(q^*) \leq f_y(q^*)$ if $p_z^* = 0$ and $f_x(q^*) \leq f_z(q^*)$ if $p_y^* = 0$. Together with (4.1) this implies that $f_x(q^*) \leq 0$. However $f_x(q) > 0$ if $p_x = \nu$ and hence $p_x^* > \nu$. It remains to consider three cases:

Case 1) $p_y^* = 0$ and $p_z^* > 0$. Then (4.2) implies that $f_x(q^*) = f_z(q^*)$ and $f_y(q^*) \leq \min\{f_x(q^*), f_z(q^*)\}$. Together with (4.1) it follows that $f_x(q^*) = f_z(q^*) \leq 0$ and hence $f_y(q^*) \leq 0$.

Case 2) $p_y^* > 0$ and $p_z^* = 0$. Observe that $z^b(q) = 0$ if $p_z = 0$. Then, analogously to case 1) it follows that $f_z(q^*) \leq f_x(q^*) = f_y(q^*) \leq 0$.

Case 3) $p_y^* = 0$ and $p_z^* = 0$. It follows from (4.1) that $f_x(q^*) \leq 0$, while (4.2) implies that $f_y(q^*) \leq f_x(q^*) \leq 0$ and $f_z(q^*) \leq f_x(q^*) \leq 0$.

This proves that $\xi(q^*) \leq 0$.

Theorem 4.6. If E is regular and under the Assumptions 4.1 - 4.3, there exists an industrial equilibrium.

Proof. Under the regularity condition we have that f_x, f_y, f_z, $f^1,\ldots,f^h$, $\tau^1,\ldots,\tau^h$, are continuous functions from $S^2_\nu \times R^{2h}(M,T)$ to R. Hence ξ is continuous and according to Lemma 4.4 has a stationary point q^*. According to lemma 4.5 we have that $\xi(q^*) \leq 0$.

It remains to prove that $q^* = (p_x^*, p_y^*, p_z^*, p^{1*},\ldots,p^{h*}, t^{1*},\ldots,t^{h*})^T$ yields an equilibrium. Firstly, from the fact that $\xi(q^*) \leq 0$, it follows that q^* the quantity constraints $x^i \leq A$, $y^i \leq B$ and $z^i \leq C$ are not binding. Hence, for all i, $x^i(q^*)$, $y^i(q^*)$, $z^i(q^*)$ maximizes i-th utility under the budget restiction

$$p_x^*x^i + (p_y^*+t^{i*})y^i + p^{i*}(z^i-\bar{z}) \leq B^i(q^*),$$

while this restriction is satisfied with equality. The latter implies that $L(q^*) = 0$. Since for all i, $p^{i*} \geq 0 \perp f^i(q^*) \leq 0$ and $t^{i*} \geq 0 \perp \tau^i(q^*) \leq 0$ this implies that $p_x{}^* f_x(q^*) + p_y{}^* f_y(q^*) + z^b(q^*) f_z(q^*) = 0$ and hence it follows that $f_x(q^*) = 0$, $p_y{}^* \geq 0 \perp f_y(q^*) \leq 0$ and $p_z{}^* \geq 0 \perp f_z(q^*) \leq 0$.

Now, let i be fixed for the moment. Suppose $p^{i*} = 0$. From the monotonicity of the preferences we then have that $z^i(q^*) = C$. This contradicts $z^i(q^*) < C$ and hence we have that $p^{i*} > 0$. From this it follows that $f^i(q^*) = z^i(q^*) - z^b(q^*) - \bar{z} = 0$ and therfore $z^i(q^*) = z^b(q^*) + \bar{z}$. So, for all i, $x^i(q^*)$, $y^i(q^*)$, $z^b(q^*) + \bar{z}$ maximizes $u^i(x^i, y^i, z^i)$ under the budget restriction

$$p_x{}^* x^i + (p_y{}^* + t^{i*}) y^i + p^{i*}(z^i - \bar{z}) \leq B^i(q^*),$$

and hence $x^i(q^*)$, $y^i(q^*)$, $z^b(q^*)$ satisfy equilibrium condition 1).

From $f_x(q^*) = \Sigma_i x^i(q^*) - \Sigma_i w^i - x^a(q^*) - x^b(q^*) = 0$ we have that the market of the private composite commodity is in equilibrium and hence $x^i(q^*)$, $i = 1,\ldots,h$, $x^a(q^*)$, $x^b(q^*)$ satisfy the first equation of equilibrium condition 4). From this it also follows that $-x^a(q^*) < A$ and $-x^b(q^*) < A$ and hence the producers maximize their profits under the technical constraints only, implying that $(x^a(q^*), y(q^*))$ and $(x^b(q^*), z(q^*))$ satisfy the profit maximization condition 2). Since $y^a(q) = 0$ if $p_y = 0$ it follows that $f_y(q) = \Sigma_i y^i(q) - y^a(q) \geq 0$ if $p_y = 0$ and hence we have that $f_y(q^*) = \Sigma_i y^i(q^*) - y^a(q^*) = 0$, so that also the second equation 2 of equilibrium condition 4) is satisfied. From $f_z(q) = \Sigma_i \{p^i + t^i \delta y^i(z^b(q))/\delta z^b] - p_z$, we obtain that $f_z(q) \geq 0$ if $p_z = 0$ and hence $f_z(q^*) = 0$, implying that $\Sigma_i \{p^{i*} + t^{i*} \delta y^i/\delta z^b(q^*)] = p_z{}^*$, so that p^{i*}, t^{i*}, $i = 1,\ldots,h$, $p_z{}^*$ and $z^b(q^*)$ satisfy the first part of condition 5). From $t^{i*} \geq 0 \perp \tau^i(q^*) \leq 0$ it follows immediately that for all i, $y^i(q^*)$, $y^i(z^b(q^*))$ and t^{i*} satisfy equilibrium condition 3).

From this latter condition it follows immediately that $t^{i*} y^i(q^*) = t^{i*} y^i(z^b(q^*))$. Hence, $\pi^0(q^*) = \Sigma_i t^{i*} y^i(z^b(q^*))(1 - \epsilon^i(q^*)) = \Sigma_i t^{i*} y^i(q^*)(1 - \epsilon^i(q^*))$. So, also the second part of condition 5) is satisfied. Hence, $q^* = (p_x{}^*, p_y{}^*, p_z{}^*, p^{1*},\ldots,p^{h*}, t^{1*},\ldots,t^{h*})$ and the corresponding $x^i(q^*)$, $y^i(q^*)$, $i=1,\ldots,h$, $x^a(q^*)$, $y^a(q^*)$, $x^b(q^*)$, $z^b(q^*)$ and $\pi^0(q^*)$ satisfy all equilibrium conditions. Because of the homogeneity of degree zero in q and since $p_x{}^* > 0$, we obtain an equilibrium with $p_x = 1$ by dividing all components of q^* by $p_x{}^*$. This proves the existence of an equilibrium.

Oberve that under the assumptions we have proved that in an equilibrium the producer price p_z equals the total consumers' contributions p^i per unit of the produced public good plus the mark-ups t^i per unit of complementary private good. This does not imply that $z^b > 0$. If the revenues are not enough to cover the marginal cost price for any extension z^b of the public infrastructure, z^b should be equal to zero.

5. Implementation and computation.

In order to calculate a point q^*, such that $\xi(q^*) \leq 0$, we can use for instance simplicial algorithms. Such algorithms first have been designed by Scarf and Kuhn for fixed point problems on the unit price simplex. **Van der Laan** and **Talman** [3] developed a variable dimension algorithm for problems on the unit simplex. Similar algorithms for fixed or zero point problems on R^n have been introduced by van der Laan and Talman, Wright, Kojima and Yamamoto, and others. Furthermore these algorithms can be adapted for problems on the product space of a unit simplex and the n-dimensional Euclidean space (see e.g. **Hofkes** [2]). The latter algorithms can be applied to find a stationary point q^* of ξ. However, applying such an algorithm on $S^2 \times R_+^{2h}$ we have to solve a (2h+2)-dimensional problem. This will take a lot of computer time. However, under some simplifying assumptions we may reduce the problem of finding an equilibrium to a one-dimensional problem, highlightening the central issue of this paper.

First, we assume, without loss of the generality of our approach, that the public good does not appear in the utility function of the consumers, i.e., $p_z{}^i=0$ for all i. In this case the consumers are only interested in the infrastructure if they are constrained. Now, the planner's problem becomes: find z such that

$$z^b \geq 0 \perp \Sigma_i\ t^i \delta y^i/\delta z^b] \leq p_z. \qquad (5.1)$$

For the firms we take constant returns to scale production functions, i.e., $F^a(x^a,y^a) = ay^a + x^a$ and $F^b(x^b,z^b) = bz^b + x^b$ for some $a > 0$ and $b > 0$. From this we obtain immediately that in equilibrium

$$y^a \geq 0 \perp p_y \leq a \text{ and } z^b \geq 0 \perp p_z \leq b$$

and

$$\pi^a = \pi^b = 0.$$

Furthermore, the assumptions $z > 0$ and $y^i(z) > 0$ have only been made for technical reasons to prove existence. Here we assume $z = 0$ and $y^i(z^b) = a^i z^b$, so that $y^i(z) = y^i(0) = 0$ and $\delta y^i/\delta z^b = a^i$. This implies that the planner's porfit is equal to zero, and hence the consumer i-th's income is equal to the values of his initial endowmwnt w^i. For the consumers we take Cobb-Douglas utility functions. Recall that we assume that the public good does not appear in these functions. So, for i=1,...,h, the utility of consumer i is given by

$$u^i(x^i,y^i) = \rho_1{}^i \ln x^i + \rho_2{}^i \ln y^i,$$

where $\rho_1{}^i+\rho_2{}^i$ is normalized to one. Maximizing utility of consumer i under the budget constraint with $p_x = 1$

$$x^i + (p_y+t^i)y^i = w^i$$

gives for the consumer's demand

$$x^i = \rho_1{}^i w^i \qquad (5.2)$$

$$y^i = \rho_2{}^i w^i/(p_y+t^i) \qquad (5.3)$$

For given z and demand y^i, the mark-up t^i is determined by the firm F^a by setting

$$t^i = \max\{0, (\rho_2{}^i w^i/a^i z^b) - p_y\}. \qquad (5.4)$$

So, the mark-ups are determined by the industry such that the individual demands do not exceed the individual constraints $a^i z^b$. From (5.3) and (5.4) we obtain that

$$y^i = \rho_2{}^i w^i/p_y \text{ and } t^i = 0 \qquad \text{if } \rho_2{}^i w^i/p_y \le a^i z^b \qquad (5.5)$$

and

$$y^i = a^i z \text{ and } t^i = \rho_2{}^i w^i/a^i z^b - p_y \qquad \text{if } \rho_2{}^i w^i/p_y > a^i z^b. \qquad (5.6)$$

Observe that the discrimination among consumers is determined by the parameters $\rho_2{}^i$, w^i and a^i. In fact, the willingness to pay increases with $\rho_2{}^i$ and w^i and decreases with a^i.

The production level y^a is set by the producer to equal total demand, i.e.,

$$y^a = \Sigma_i\, y^i. \qquad (5.7)$$

Consequently, given the infrastructure level z^b, and assuming positive production so that $p_y = a$ and $p_z = b$, the values of all other variables, namely quantities and mark-ups, can be calculated through (5.1)-(5.7). So, the equilibrium problem is to find a level z of the infrastructure such that the markets for the composite commodity clears and the mark-up revenues are equal to the costs of the infrastructure, i.e.,

$$\Sigma_i\, x^i = \Sigma_i\, w^i + x^a + x^b \qquad \text{(market condition)}$$

$$\Sigma_i\, a^i t^i = p_z = b \qquad \text{(planner condition).}$$

Since the budget conditions are satisfied with equality it follows that the market condition holds if the plannner condition is satisfied. So, substituting (5.4) in the planner condition we find an equilibrium by solving the equation

$$\Sigma_i\, a^i \max\{0, (\rho_2{}^i w^i/a^i z^b) - p_y\} = b$$

in the variable z^b.

This highlights the issue of the paper. The problem of the optimal level of the infrastrucuture can be solved by the planner (or industry) if the industry is able to calculate the mark-ups on the use of the complementary private good.

This paper has been concerned with the problem of financing an infrastructure needed for operating and utilizing private services and commodities.

The paper has to be seen as a first attempt to give a solid framework for the idea that the industry plays a central role in financing the infrastructure. In fact, the infrastructure is financed through mark-ups on the private services and commodities that make use of it. These mark-ups come from the constraints experienced by the agents. With respect to the consumers, the level of the infrastructure yields a (subjective) constraint on their private consumption. In case of producers the level of the infrastructure puts a constraint on their production possibilities. The mark-ups reveal these constraints and therefore the need for the infrastructure. Given the mark-ups the agents are willing to pay, the planner determines the optimal level of the infrastructure. In a next paper we want to consider the problem of determining the mark-ups. We want to elaborate the idea that the individual mark-ups are determined by the industry and are corporated in the prices the producers set for their products. We may think of a partitioning of the consumers into a number of groups. Then for each group the industry sets the mark-ups by considering a representative agent. So, in this way we get different prices for different types of agents.

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