Integrable Systems and Symplectic Geometry

Asadi, E.

2008

Summary

Many of the equations and systems which now are called integrable have been known in differential geometry. One of them is the famous sine-Gordon equation which was derived to describe the pseudospherical surfaces. The Bäcklund transformation generates an infinite-dimensional 'symmetry group' acting on the set of pseudospherical surfaces and the permutability theorem of Bianchi, shows the possibility of writing down explicit solutions starting with a simple surface. Another one is the Liouville equation describing minimal surfaces in Euclidean space. For physicists, the prototype examples of integrable systems are the Korteweg-De Vries equation and the nonlinear Schrödinger equation.

A starting point from which all this rich structure can be derived is a zero-curvature formulation of the underlying problem or the Lax representation of nonlinear equations. The zero curvature representation has a transparent geometrical origin. In differential geometry, the embedded surface is the Gauss-Codazzi equation represented as a compatibility condition of linear equations for the moving frame.

The connection between geometry and integrable systems is clarified by Hasimoto in 1972. He found the transformation between the equations governing the curvature and torsion of a thin vortex filament moving in a fluid and the NLS equation. In fact Hasimoto constructed the complex function of the curvature and torsion of the curve and showed that if the curve evolves according to the vortex filament equation, then this function solves the cubic nonlinear Schrodinger equation. One can find, through the Hasimoto transformation, the recursion operator for NLS hierarchy as well. Later is showed that the Hasimoto transformation is induced by a gauge transformation from the Frenêt frame to the parallel or natural frame.

Generalizing these result to the motion of a curve in the Riemannian manifold with constant curvature following an arc-length preserving geometric evolution, gives rise to the evolution of its curvature and torsion which proved to be Hamiltonian flow. By using the parallel frame, one can find the recursion operator as well as the Hamiltonian and Symplectic operators. This can be done equivaletly by Cartan structure equation having a Cartan connection which is specified according to the frame we choose. Similarly this method can be used in conformal geometry as well.

In this thesis, we consider the symplectic geometry defined by the homogeneous space which indeed is identified with projective quaternionic space. We study the Cartan

structure equation, and see that choosing natural or parallel frame, one can find the time evolution of invariants of a family of curves embedded in the homogeneous space, the recursion operator, Nijenhuis operator as well as the Hamiltonian and symplectic operators. Replacing the unknown variables with trivial symmetries, one can get an noncommutative integrable system. We also express all those operators in terms of Lie bracket, Killing form and projections on the underlying subspaces. The method is employed is enough general to say that choosing a "right frame" would lead the Cartan structure equation to integrable equations together with all geometric operators.

Generalizing the Drinfel'd-Sokolov method to the symplectic geometry, we find the Lax representation of the equation we found.