Integrable Systems and Symplectic Geometry

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2008

Introduction

Some history

Many of the equations and systems which now are called integrable have been known in differential geometry. One of them is the famous sine-Gordon equation (SG), which was derived to describe surfaces with constant negative Gaussian curvature. Another one is the Liouville equation describing minimal surfaces in $3-$dimensional Euclidean space. For physicists, the prototype examples of integrable systems are the Korteweg-De Vries equation (KDV) [15] and the nonlinear Schrödinger equation (NLS) [19].

"***what is an integrable system?***"

This is a question with many answers of varying degree of precision, generality and plausibility. We will try briefly to list few of these answers.

Newton's equations of motion are those three famous equations which are taught to every student of elementary Classical Mechanics. The Kepler two-body problem and a few other equations turned out to have "exact solutions".

In fact these equations are particular cases of more general mechanical systems, known as (finite dimensional) Hamiltonian system, with a Hamiltonian function and a Poisson bracket.

A Hamiltonian system is called "completely integrable" if it has as many independent functions in involution with the Hamiltonian function and themselves as it has degrees of freedom.

In the nineteenth century, Liouville provided a general framework characterizing the cases where completely integrable Hamiltonian system are "solvable by quadratures", i.e., the general solution is found by integration and algebraic operations only, see [3].

The discovery of the physical soliton is attributed to John Russell's observation in 1834 as he described it in his "Report on Waves" [55]. Much later in 1895 Korteweg and de Vries derived the equation for water waves in shallow channels, which confirms the existence of solitary waves. The equation which now bears their names is of the form

$$u_t = u_3 + uu_1 \quad (\textbf{KDV equation}).$$

In 1965, Kruskal and Zabusky, following a computationally numerical study of the Boussinesq anharmonic lattice of equal masses which was done by Fermi, Pasta and Ulam (FPU), rederived the KDV equation and found its stable pulse-like waves. They named such waves **solitons**. These are solitary waves in the form of pulses whose behavior has many particle-like features. During their evolution, solitons propagate without change of shape and with no energy loss. When two or more solitons with different propagation speed collide, after a highly nonlinear interaction the pulses emerge with the same initial form and no energy is lost in radiation in the course of the interaction.

The stability and particle-like behavior of the solitons could only be explained by the existence of many conservation laws : $D_tU + D_xF = 0$; in which U is called **conserved density** and F **conserved flux**. Zabusky and Kruskal started to find more of them. Later, it was proved by Miura, Gardner and Kruskal in 1968 that there was indeed a conserved density of each order [50].

Gardner was the first to notice that the KDV equation could be written in a Hamiltonian framework. Later Zakharov and Fadeev showed how this could be interpreted as a completely integrable Hamiltonian system in a same sense as finite dimensional integrable systems [78] where one finds for each degree of freedom a conserved density.

Perhaps the richest group of equations known to be integrable are pseudospherical surfaces. They are surfaces in $\mathbb{R}^3$ with constant negative Gaussian curvature. Bianchi [18] discovered a beautiful relation between iterated Bäcklund transformations: the permutability theorem. This theorem assert that for two Bäcklund transformations $f_1 = \mathcal{B}_{\sigma_1} f$ and $f_2 = \mathcal{B}_{\sigma_2} f$ of a pseudospherical surface f corresponding to angles σ_1, σ_2 between the normals, there exists a fourth pseudospherical surface $\hat{f}$ which is simultaneously a Bäcklund transformation of f_1 and f_2 :

$$\hat{f} = \mathcal{B}_{\sigma_1} f_2 = \mathcal{B}_{\sigma_2} f_1.$$

Moreover $\hat{f}$ can be computed algebraically from f, f_1, f_2. In this way, the Bäcklund transformation generates an infinite-dimensional 'symmetry group' acting on the set of pseudospherical surfaces and the permutability theorem shows the possibility of writing down explicit solutions starting with a simple f.
We might say that the symmetry of an equation is the conserved geometric feature of solitons. The symmetry groups of differential equations were first studied by Sophus Lie. In his framework, these consist of geometric transformations of independent and dependent variables of the system. In the case of KDV, there are four such symmetries, namely arbitrary translation in x and t, Galilean boost and scaling. In the context of pseudospherical surfaces, see [6], Bäcklund transformation $\mathcal{B}_\sigma$ is the transform of a Bianchi transformation by means of a Lie transformation L_σ, symbolically

$$\mathcal{B}_\sigma = L_\sigma^{-1} \mathcal{B}_{\pi/2} L_\sigma. \qquad (0.0.1)$$

As we explained, we get the following features of an integrable system:

1. infinitely many generalized symmetries;
2. infinitely many conservation laws;
3. explicit solutions;
4. complete integrability in the sense of Liouville.

Where does integrability come from?

A starting point from which all this rich structure can be derived is a zero-curvature formulation of the underlying problem. The Lax (or Zakharov-Shabat (ZS)) representation of nonlinear equation can be given in a form of compatibility condition

$$U_t(\lambda) - V_x(\lambda) + [U(\lambda), V(\lambda)] = 0 \tag{0.0.2}$$

of two linear equations

$$\Psi_x = U(\lambda)\Psi, \quad \Psi_t = V(\lambda)\Psi.$$

See [22] for the reduction of this construction in symmetric space. In that case, $U = \lambda A + Q(x,t)$ in which A is constant element of the underlying Lie algebra and Q is potential function. In this way, Zakharov and Mikhailov [77] use a pole expansion $U = \sum U_(q)(\lambda - \lambda_i)^{-1}$, while others [35, 38] favor polynomial expansions.

The zero curvature representation (0.0.2) has a transparent geometrical origin. In differential geometry, the embedded surface is the Gauss-Codazzi equation represented as a compatibility condition of linear equations for the moving frame (the Gauss-Weingarten equations), see Lund and Regge [47]. The spectral parameter λ in this representation describes deformation of surfaces preserving their properties.

The connection between geometry and integrable systems is clarified by Hasimoto [32] in 1972. He found the transformation between the equations governing the curvature and torsion of a thin vortex filament (FM) moving in an incompressible inviscid fluid and the NLS equation. The equation FM can be modeled as

$$\gamma_t = \gamma_s \times \gamma_{ss}$$

in which $\gamma(s,t)$ is a curve evolving in $3-$dimensional space $\mathbb{R}^3$. In fact Hasimoto constructed the complex function $\psi = \kappa \exp(i \int_0^s \tau ds)$ of the curvature and torsion of the curve γ, and showed that if the curve evolves according to the vortex filament equation, then ψ solves the cubic nonlinear Schrodinger equation

$$i\psi_t + \psi_{ss} + \frac{1}{2}|\psi|^2\psi = 0.$$

Lamb [41] used the Hasimoto transformation to connect other motion of curves to the integrable equations like modified KDV (mKDV) and SG equations. Balakrishnan et al. [5] have investigated another aspect of space curve formulation: the geometric phase associated with the time evolution of the curve and its connection to integrability.

Sasaki [62] gave a geometric interpretation of the ZS spectral problem in terms of pseudospherical surfaces. Chern and Tenenblat [10] characterized the mKDV hierarchy as a relation between local invariants of a certain foliation on a surface of constant nonzero Gauss curvature. Terng, Tenenblat, Sattinger and Uhlenbeck in a series of papers [69, 69, 70, 68, 67], studied the symplectic, Lie theoretic, and differential geometric properties of soliton theory. They construct a pencil of connections depending on the deformation parameter λ, and prove that the pencil is flat for all $\lambda \in \mathbb{R}$ if and only if the dynamical variables or the invariance of the one parameter of surfaces follows a Hamiltonian flow. See also [9, 8].

Langer and Perline [43] showed in 1991 that the dynamics of a nonstretching vortex filament in $\mathbb{R}^3$ gives rise, through the Hasimoto transformation, to the recursion operator of the NLS hierarchy. The appearance of the recursion operator can be explained observing that the Frenêt equations for the curve in $\mathbb{R}^2$ and $\mathbb{R}^3$ are equivalent to the ZS spectral problem without the spectral parameter.

Doliwa and Santini [12] showed that certain elementary geometric properties of the motion of a curve select the hierarchy of integrable dynamics. The motion should be nonstretching and occur in a $N-$dimensional sphere of radius R and the dynamics independent of the radius of the sphere. They give a simple geometric meaning of the Hasimoto transformation: Hasimoto transformation is induced by a gauge transformation from the Frenêt frame to the parallel or natural frame. Wang [73] uses this interpretation to find the generalized (from $\mathbb{R}^3$ to $\mathbb{R}^N$) Hasimoto transformation.

Generalizing Doliwa and Santini's approach, Sanders, Wang and Beffa showed that motion of a curve in a $3-$dimensional Riemannian manifold with constant curvature follows an arc-length preserving geometric evolution and the evolution of its curvature and torsion is always a Hamiltonian flow.

Cartan's Lemma leads us to use the Lie algebra valued $1-$form instead of the Levi-Civita connection defined on Riemannian manifold, so that having a frame on the curve embedded in the Riemannian manifold is equivalent to specifying the Cartan connection applied on the γ_s. Indeed Sanders and Wang [58] showed that choosing a natural frame and having the Cartan connection specified according to the natural frame, the Cartan structure equation leads to the recursion equation of integrable equation. In this way they found the Hamiltonian operator out of curvature part and symplectic operator resulted from solving the free torsion tensor. Authors applied a similar method to the case of conformal geometry [59], in which case making the proper choice of "natural frame", leads to the Hamiltonian and symplectic operator.

Outline and Summary of results

In this thesis, we generalized the former idea to other geometries, such as $\mathfrak{o}(p,q)-$orthogonal geometry and mainly to symplectic geometry.

Chapter 1 is an introduction to the algebra of quaternions and the symplectic Lie algebra using quaternions.

Chapter 2 explains the variational calculus and in particular defines Hamiltonian and symplectic operators suitable for Lie algebraic domains.

Chapter 3 is an introduction to differential geometry.

Chapter 4 is the core of the thesis. Starting from Riemannian geometry in Section 4.1 we prove that if we choose the natural moving frame for a flow of a curve preserving arclength and embedded in the Riemannian manifold with Levi-Civita connection compatible with its metric, then evolution of the differential invariants of the curve follows the vector mKDV equation and gives us the recursion operator to produce higher symmetries as well as the Hamiltonian and symplectic operator. Using Cartan's Lemma we see that these objects can be obtained by just writing down the Cartan structure equation for Euclidean geometry.

We then proceed with $\mathfrak{o}(p,q)$-orthogonal geometry in Section 4.2, generalizing the Euclidean geometry, choosing the natural moving frame for the connection matrix . The Cartan structure equations lead to the evolution equation

$$D_t\mathbf{u} = -I^1_{p-1,q}\mathbf{u}_{3x} - \frac{3}{2}\mathbf{u}_x < \mathbf{u},\mathbf{u} >, \quad I^1_{p-1,q} = \begin{pmatrix} I_{p-1} & 0 \\ 0 & -I_q \end{pmatrix}$$

which is an mKDV type equation with recursion operator $\mathfrak{R} = \mathfrak{H}\mathfrak{I}$ where the operator $\mathfrak{H}$ and $\mathfrak{I}$ are proved to be the Hamiltonian and symplectic operator, respectively.

Next in Section 4.3, we consider the symplectic geometry defined by the homogeneous space $Sp(n)/Sp(1)\times Sp(n-1)$, which indeed is identified with projective quaternionic space $\mathbb{HP}^n$. We study the Cartan structure equation, and see that choosing natural or parallel frame

$$\hat{u} =: \omega(D_x) = \begin{pmatrix} 0 & 1 & \mathbf{0}^T \\ -1 & u & -\overline{\mathbf{u}}^T \\ \mathbf{0} & \mathbf{u} & \mathbf{0} \end{pmatrix},$$

one can find the time evolution of invariants of a family of curves embedded in the homogeneous space. That is,

$$\begin{pmatrix} u \\ \mathbf{u} \end{pmatrix}_t = \mathfrak{H}\mathfrak{I}\begin{pmatrix} v \\ \mathbf{v} \end{pmatrix} + \mathfrak{A}\begin{pmatrix} v \\ \mathbf{v} \end{pmatrix}.$$

Replacing v by trivial symmetry u_x, we obtain a noncommutative scalar-vector mKDV equation:

$$\begin{cases} u_t &= \frac{1}{4}u_3 + \frac{3}{8}(-uu_1u - uu_2 + u_2u) + \frac{3}{2}\langle\mathbf{u},\mathbf{u}\rangle u_1 + \langle\mathbf{u},\mathbf{u}_1\rangle u + \tfrac{1}{2}u\langle\mathbf{u},\mathbf{u}_1\rangle \\ & \quad +2u\langle\mathbf{u}_1,\mathbf{u}\rangle - \tfrac{1}{2}\langle\mathbf{u}_1,\mathbf{u}\rangle u + \frac{3}{2}C_{\mathbf{u}}\mathbf{u}_2, \\ \mathbf{u}_t &= \mathbf{u}_3 + \frac{3}{2}\mathbf{u}_2 u + \frac{3}{4}\mathbf{u}_1(u_1 + \frac{1}{2}u^2 + 2\langle\mathbf{u},\mathbf{u}\rangle). \end{cases}$$

This is equation 4.3.10. The reduction $\mathbf{u} = 0$ leads to the second version of the noncommutative mKDV scalar equation and the reduction $u = 0$ yields the vector mKDV equation.

Then in Section 4.4 one rewrites this equation using the Lie bracket, Killing form and projections. In the symplectic case, we will see that

$$\hat{u}_t = \hat{\mathfrak{H}}\hat{\mathfrak{I}}\hat{v} + \hat{\mathfrak{A}}\hat{v},$$

in which

$$\begin{aligned}
\hat{\mathfrak{H}} &= D_x - \pi_1 \mathrm{ad}_{\hat{u}} - \mathrm{ad}_{\hat{u}} D_x^{-1} \pi_0 \mathrm{ad}_{\hat{u}}, \\
\hat{\mathfrak{I}} &= -\frac{1}{2}\hat{u} D_x^{-1} K(\hat{u}, .) - (\frac{1}{2}\rho_1 + \rho_0)\pi_1 \mathrm{ad}_{\hat{a}} (D_x - \mathrm{ad}_{\hat{u}}) \mathrm{ad}_{\hat{a}} \pi_1 (\frac{1}{2}\rho_1 + \rho_0), \\
\hat{\mathfrak{A}} &= \rho_0 + 2\rho_1 - \mathrm{ad}_{\hat{u}} D_x^{-1} \rho_1.
\end{aligned}$$

This way of writing the geometric operator can be generalized to any other Cartan geometry.

We prove that the operator $\hat{\mathfrak{A}}$ is Nijenhuis operator, that is, the Nijenhuis tensor vanishes. Furthermore we claim that $\hat{\mathfrak{H}}$ and $\hat{\mathfrak{H}}\mathfrak{A}^*$ are Hamiltonian operators and $\hat{\mathfrak{A}}^{-1*}\hat{\mathfrak{I}}\hat{\mathfrak{A}}^{-1}$ is symplectic. The proofs can be found in Chapter 6

Generalizing the Drinfel'd-Sokolov method to symplectic geometry, we find the Lax representation of the symplectic case in Chapter 5. The Lax operator is indeed

$$L = D_x + \lambda A + q,$$

in which A is the projection of $\omega(D_x)$ to the vector space $\mathfrak{sp}(n)/\mathfrak{sp}(1) \times \mathfrak{sp}(n-1)$ as a constant element of the symplectic Lie algebra and q is the projection of $\omega(D_x)$ to the subalgebra $\mathfrak{sp}(1) \times \mathfrak{sp}(n-1)$.