Integrable Systems and Symplectic Geometry

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Chapter 1

Quaternionic algebra

1.1 Introduction

In this chapter we consider quaternionic numbers and vectors and see how an inner product can be defined on the space of quaternionic vectors. We introduce the Lie algebra of quaternions and compute the Killing form of two elements of this Lie algebra. We then find what can be the relation between the inner product of two vectors in quaternionic space and the Killing form of two specific elements of the Lie algebra.

Let us first briefly recall some basic facts about quaternions. The quaternions were discovered on 1843 by Sir William Rowan Hamilton. They form a noncommutative, associative algebra over $\mathbb{R}$:

$$\mathbb{H} = \{a + bi + cj + dk \,|\, a, b, c, d \in \mathbb{R}\},$$

where algebra multiplication is defined as

$$i^2 = j^2 = k^2 = -1, \quad ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik,$$

and scalar multiplication defined as

$$\alpha(a + bi + cj + dk) = (\alpha a) + (\alpha b)i + (\alpha c)j + (\alpha d)k \quad \text{for} \quad \alpha \in \mathbb{R}.$$

In general, by associative algebra over $\mathbb{R}$, we mean a vector space over the field $\mathbb{R}$ with a multiplication on it which is associative, distributive over addition and satisfies

$$\alpha(q_1 q_2) = q_1(\alpha q_2) = (\alpha q_1)q_2 \quad \text{for} \quad \alpha \in \mathbb{R} \quad \text{and} \quad q_1, q_2 \in \mathbb{H}.$$

Moreover $\mathbb{H}$ is an involutive algebra, i.e, there is a map $^* : \mathbb{H} \to \mathbb{H}$ such that $(uv)^* = v^* u^*$ and $u^{**} = u$, for $u, v \in \mathbb{H}$. The involution of $q \in \mathbb{H}$ is conjugation of quaternionic number $q = a + bi + cj + dk$, defined by

$$\overline{q} = a - bi - cj - dk.$$

Clearly for all $u, v \in \mathbb{H}$ we have that $\overline{uv} = \overline{v}\,\overline{u}$. We define the *modulus* of a quaternion q by

$$|q| = (q\overline{q})^{1/2} = (a^2 + b^2 + c^2 + d^2)^{1/2}.$$

The principal properties of the modulus of $\mathbb{H}$ are as follows. If $q, q_1, q_2 \in \mathbb{H}$, then

1. $|q| = 0$ if and only if $q = 0$;
2. $|q_1 + q_2| \leq |q_1| + |q_2|$ and $|q_1 q_2| = |q_1||q_2|$;
3. $|\overline{q}| = |q|$; and
4. If $q \neq 0$, then $q^{-1}q = qq^{-1} = 1$, where $q^{-1} = \dfrac{\overline{q}}{|q|^2}$.

Observe that the set $\mathbb{C}$ of complex numbers appears as a real subalgebra of $\mathbb{H}$, meaning that $\mathbb{C}$ can be embedded in $\mathbb{H}$ as an associative algebra over $\mathbb{R}$. More precisely $\mathbb{C}$ can be seen as $Span_{\mathbb{R}}\{1, i\}$ residing inside $\mathbb{H}$ where $Span_{\mathbb{R}}\{1, i\} = \{a1 + bi \mid a, b \in \mathbb{R}\}$ is spanned by $1, i$ with coefficients in $\mathbb{R}$. The following result gives some of the properties of the subalgebra $\mathbb{C}$ in $\mathbb{H}$.(The proofs are elementary and omitted.)

Lemma 1.1.1. Consider $\mathbb{C}$ as the real subalgebra $Span_{\mathbb{R}}\{1.i\}$ of $\mathbb{H}$. Then:

1. $Span_{\mathbb{R}}\{1, i\} = \{q \in \mathbb{H} \mid qi = iq\}$;
2. $Span_{\mathbb{R}}\{j, k\} = \{q \in \mathbb{H} \mid qi = -iq\}$;
3. $\lambda q = q\overline{\lambda}$ for every $\lambda \in Span_{\mathbb{R}}\{1, i\} = \mathbb{C}$ and $q \in Span_{\mathbb{R}}\{j, k\}$.

Remark 1.1.2. 1. The first point to be emphasized in the third item of this lemma is that if $\lambda \in \mathbb{C}$, then $\overline{\lambda}$ can be obtained from λ by an similarity transformation $\lambda \to q^{-1}\lambda q$ in $\mathbb{H}$.(This is impossible in $\mathbb{C}$ except for the trivial case where $\lambda \in \mathbb{R}$.)

2. It follows that $\mathbb{H}$ is not an algebra over $\mathbb{C}$, since for $\lambda \in \mathbb{C}$ and $q_1, q_2 \in \mathbb{H}$, two quantities $q_1(\lambda q_2)$ and $(\lambda q_1)q_2$ are not necessarily equal.

We now derive two representation of quaternions by complex vector and matrices as well as real matrices. The identification of $\mathbb{C}$ within $\mathbb{H}$ affords a useful linear representation of quaternions, as well as $n-$tuples of quaternions, by complex vectors. Namely, if $q = a_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k \in \mathbb{H}$, then

$$q = (\alpha_0 + \alpha_1 i) + (\alpha_2 + \alpha_3 i)j = \gamma_1 + \gamma_2 j \tag{1.1.1}$$

where $\gamma_1 = \alpha_0 + \alpha_1 i$ and $\gamma_2 = \alpha_2 + \alpha_3 i$ belong to $\mathbb{C}$. Thus, each quaternion q is uniquely represented by a pair of complex numbers γ_1, γ_1 via (1.1). In fact the function $\rho : \mathbb{H} \to \mathbb{C}^2$ defined by

$$\rho(q) = \begin{pmatrix} \gamma_1 \\ -\overline{\gamma_2} \end{pmatrix}$$

is a linear one-to-one map between $\mathbb{H}$ and $\mathbb{C}^2$ as vector spaces over $\mathbb{R}$. This representation extends to the real vector space $\mathbb{H}^n$ of $n-$tuples of quaternions. Indeed, define the function

$$\rho : \mathbb{H}^n \to \mathbb{C}^{2n} \quad \text{by} \quad \rho(\xi) = \begin{pmatrix} \xi_1 \\ -\overline{\xi}_2 \end{pmatrix},$$

where $\xi = \xi_1 + \xi_2 j \in \mathbb{H}^n$ and $\xi_1, \xi_2 \in \mathbb{C}^n$. A second useful representation of quaternions is via $M(2,\mathbb{C})$ as the set of all 2×2 complex matrices. Consider the function $\phi : \mathbb{H} \to M(2,\mathbb{C})$ defined as follows: if $q = \gamma_1 + \gamma_2 j$, as in (1.1), then

$$\phi(q) = \begin{pmatrix} \gamma_1 & \gamma_2 \\ -\overline{\gamma_2} & \overline{\gamma_1} \end{pmatrix}.$$

The function ϕ is simply injective and satisfies, for all $q, q_1, q_2 \in \mathbb{H}$ and $\alpha \in \mathbb{R}$, the equations

$$\phi(q_1 + q_2) = \phi(q_1) + \phi(q_2), \quad \phi(q_1 q_2) = \phi(q_1)\phi(q_2), \quad \phi(\alpha q) = \alpha\phi(q).$$

Moreover,

$$\phi(\overline{q}) = \phi(\overline{\gamma_1} + (-\gamma_2)j) = \begin{pmatrix} \overline{\gamma_1} & -\gamma_2 \\ \overline{\gamma_2} & \gamma_1 \end{pmatrix} = \phi(q)^*$$

where $\phi(q)^*$ denotes the conjugate transpose of the 2×2 complex matrix $\phi(q)$. Thus ϕ preserves the linear, multiplicative, and involutive structure of $\mathbb{H}$; that is, ϕ is an injective $*$-homomorphism from the real involutive algebra $\mathbb{H}$ into the real involutive algebra $M(2,\mathbb{C})$.

A homomorphic embedding of $M(n,\mathbb{H})$ into $M(2n,\mathbb{C})$ is similarly constructed. First, note that by applying (1.1) entry wise to a matrix $Q \in M(n,\mathbb{H})$, one can write Q as $Q = \Gamma_1 + \Gamma_2 j$, where Γ_1 and Γ_2 are $n \times n$ complex matrices. The function $\Phi : M(n,\mathbb{H}) \to M(2n,\mathbb{C})$, defined for $Q = \Gamma_1 + \Gamma_2 j \in M(n,\mathbb{H})$ by

$$\Phi(Q) = \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ -\overline{\Gamma_2} & \overline{\Gamma_1} \end{pmatrix}$$

is an injective $* -$ *homomorphism*. That is, Φ satisfies, for all $Q, Q_1, Q_2 \in M(n,\mathbb{H})$ and $\alpha \in \mathbb{R}$, the equations:

1. $\Phi(Q_1 + Q_2) = \Phi(Q_1) + \Phi(Q_2)$ and $\Phi(Q_1 Q_2) = \Phi(Q_1)\Phi(Q_2)$,
2. $\Phi(\alpha Q) = \alpha\Phi(Q)$, and
3. $\Phi(Q^*) = \Phi(Q)^*$ where Q^* is conjugate transpose of matrix $Q \in M(n,\mathbb{H})$, that is $Q^* = \overline{Q}^t$.

Observe that for $Q_1, Q_2 \in M(n,\mathbb{H})$, we have $(Q_1 Q_2)^* = Q_2^* Q_1^*$ either by direct reasoning or using similar identity for complex matrices and properties above of the function Φ.

Proposition 1.1.3. Left quaternionic inverse is also right inverse for quaternionic matrices.

Proof. If $Q_1 Q_2 = I$, then $\Phi(Q_1)\Phi(Q_2) = I$. Hence also $I = \Phi(Q_2)\Phi(Q_1) = \Phi(Q_2 Q_1)$. Now Φ is injective and it follows that $Q_2 Q_1 = I$. ■

Proposition 1.1.4. The image of the function Φ is

$$\Phi(M(n,\mathbb{H})) = \{P \in M(2n,\mathbb{C}) \mid \hat{J}P = P\hat{J}\}$$

where $\hat{J} = Jj$, with

$$J = \begin{pmatrix} 0 & -\sigma I_n \\ \sigma I_n & 0 \end{pmatrix}.$$

Proof. We see that $\Phi(I_n j) = J$, and if $\Phi(Q) = P$ where $Q = \Gamma_1 + \Gamma_2 j$, then

$$JP = \Phi(I_n j)\Phi(Q) = \Phi(I_n jQ) = \Phi(jQ).$$

Using Lemma 1.1.1, part 3, we see that

$$jQ = j\Gamma_1 + j\Gamma_2 j = \overline{\Gamma_1}j + (\overline{\Gamma_2}j)j = \overline{\Gamma_1}j - \overline{\Gamma_2}.$$

Hence $JP = \Phi(\overline{\Gamma_1}j - \overline{\Gamma_2})$. On the other hand, from the definition we obtain that

$$\overline{P} = \begin{pmatrix} \overline{\Gamma_1} & -\sigma\overline{\Gamma_2} \\ \sigma\overline{\overline{\Gamma_2}} & \Gamma_1 \end{pmatrix} = \begin{pmatrix} \overline{\Gamma_1} & -\sigma\overline{\Gamma_2} \\ \sigma\Gamma_2 & \Gamma_1 \end{pmatrix} = \Phi(\overline{\Gamma_1} + \overline{\Gamma_2}j).$$

Consequently we get that

$$\overline{P}J \quad = \quad \Phi(\overline{\Gamma_1} + \overline{\Gamma_2}j)\Phi(I_n j) = \Phi((\overline{\Gamma_1} + \overline{\Gamma_2}j)I_n j) = \Phi(\overline{\Gamma_1}j - \overline{\Gamma_2}).$$

Hence $JP = \overline{P}J$ and hence $\hat{J}P = JjP = J\overline{P}j = PJj = P\hat{J}$. On the other hand, if $\hat{J}P = P\hat{J}$, then $JP = \overline{P}J$. Hence let P be $P = \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_3 & \Gamma_4 \end{pmatrix}$, then we obtain that

$$\Gamma_4 = \overline{\Gamma_1}, \quad \Gamma_3 = -\overline{\Gamma_2}.$$

This indeed, by definition, means that $\Phi(\Gamma_1 + \Gamma_2 j) = P$. ∎

The following Lemma shows how the function $\rho : \mathbb{H}^n \to \mathbb{C}^{2n}$ is related to the function $\Phi : M(n,\mathbb{H}) \to M(2n,\mathbb{C})$.

Lemma 1.1.5. Assume that $Q \in M(n,\mathbb{H}), \xi \in \mathbb{H}^n$, and $\lambda \in \mathbb{C}$. Then:

1. $\rho(Q\xi) = \Phi(Q)\rho(\xi)$,
2. $\rho(\xi\lambda) = \lambda\rho(\xi)$, and
3. $\Phi(Q)\rho(\xi) = \lambda\rho(\xi)$ if and only if $Q\xi = \xi\lambda$.

Proof. Express Q and ξ in linear form: $Q = \Gamma_1 + \Gamma_2 j$ and $\xi = \xi_1 + \xi_2 j$, for some $\Gamma_1, \Gamma_2 \in M(n,\mathbb{C})$ and $\xi_1, \xi_2 \in \mathbb{C}^n$. Thus,

$$\begin{aligned} Q\xi &= (\Gamma_1 + \Gamma_2 j)(\xi_1 + \xi_2 j) \\ &= \Gamma_1\xi_1 + \Gamma_1\xi_2 j + \Gamma_2(j\xi_1) + \Gamma_2 j(\xi_2 j) \\ &= \Gamma_1\xi_1 + \Gamma_1\xi_2 j + \Gamma_2\overline{\xi_1}j + \Gamma_2\overline{\xi_2}j^2 \\ &= (\Gamma_1\xi_1 - \Gamma_2\overline{\xi_2}) + (\Gamma_1\xi_2 + \Gamma_2\overline{\xi_1})j, \end{aligned}$$

using part 3 of Lemma 1.1.1. Hence by definition we obtain that

$$\begin{aligned}\rho(Q\xi) &= \begin{pmatrix} \Gamma_1\xi_1 - \Gamma_2\overline{\xi_2} \\ \\ -\overline{\Gamma_1\xi_2 + \Gamma_2\overline{\xi_1}} \end{pmatrix} = \begin{pmatrix} \Gamma_1\xi_1 - \Gamma_2\overline{\xi_2} \\ \\ -\overline{\Gamma_1}\,\overline{\xi_2} - \overline{\Gamma_2}\xi_1 \end{pmatrix} \\ &= \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ \\ -\overline{\Gamma_2} & \overline{\Gamma_1} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \\ -\overline{\xi_2} \end{pmatrix} = \Phi(Q)\rho(\xi),\end{aligned}$$

which completes the proof of (1).

To prove (2), note that $\xi\lambda = (\xi_1 + \xi_2)\lambda = \xi_1\lambda + \xi_2 j\lambda = (\xi_1\lambda) + (\xi_2\overline{\lambda})j$ (by Lemma 1.1.1), and so

$$\rho(\xi\lambda) = \begin{pmatrix} \xi_1\lambda \\ -\overline{\xi_2\overline{\lambda}} \end{pmatrix} = \lambda \begin{pmatrix} \xi_1 \\ -\overline{\xi_2} \end{pmatrix} = \lambda\rho(\xi),$$

which proves (2). It might be remarked that $\lambda\xi = \lambda\xi_1 + (\alpha\xi_2)j$, hence $\rho(\lambda\xi) = \begin{pmatrix} \lambda\xi_1 \\ -\overline{\lambda}\,\overline{\xi_2} \end{pmatrix}$ which is not equal to $\lambda\rho(\xi)$ nor $\rho(\xi)\lambda$. To prove (3), first assume that $Q\xi = \xi\lambda$. Apply ρ and obtain $\rho(Q\xi) = \rho(\xi\lambda)$. But this equation is, by (1) and (2), precisely $\Phi(Q)\rho(\xi) = \lambda\rho(\xi)$. Conversely, assume that $\Phi(Q)\rho(\xi) = \lambda\rho(\xi)$. Then, by (1) and (2), $\rho(Q\xi) = \rho(\xi\lambda)$. Because ρ is injective, we have that $Q\xi = \xi\lambda$, thereby proving (3). ■

For further study about quaternions, see [20].

Now we will discuss the identification of quaternionic matrices with real matrices. Since we can write $Q \in M(n, \mathbb{H})$ uniquely as

$$Q = \Gamma_0 + \Gamma_1 i + \Gamma_2 j + \Gamma_3 k$$

where $\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3$ are real matrices, The map $\mu : M(n, \mathbb{H}) \to M(4n, \mathbb{R})$ given by

$$\mu(Q) = \begin{pmatrix} \Gamma_0 & -\Gamma_1 & -\Gamma_2 & -\Gamma_3 \\ \Gamma_1 & \Gamma_0 & -\Gamma_3 & \Gamma_2 \\ \Gamma_2 & \Gamma_3 & \Gamma_0 & -\Gamma_1 \\ \Gamma_3 & -\Gamma_2 & \Gamma_1 & \Gamma_0 \end{pmatrix}$$

is an injective homomorphism of two real associative algebras over real numbers. That means that if $Q, Q_1, Q_2 \in M(n, \mathbb{H})$ and $\alpha \in \mathbb{R}$, then:

$$\mu(Q_1 + Q_2) = \mu(Q_1) + \mu(Q_2), \quad \mu(Q_1 Q_2) = \mu(Q_1)\mu(Q_2), \quad \mu(\alpha Q) = \alpha\mu(Q).$$

It is also clear that

$$\mu(Q^*) = \mu(\Gamma_0^t - \Gamma_1^t i - \Gamma_2^t j - \Gamma_3^t k) = \mu(Q)^t.$$

1.2 Inner product on quaternionic vectors

The set of all $n-$tuples of quaternions $\mathbb{H}^n$ is a right module over the division ring $\mathbb{H}$. we call $\mathbb{H}^n$ a *right* $\mathbb{H}-$*module*. That means, for all $q, q_1, q_2 \in \mathbb{H}$ and $\xi, \xi_1, \xi_2 \in \mathbb{H}^n$ we have that

1. $(\xi_1 + \xi_2)q = \xi_1 q + \xi_2 q$,
2. $\xi(q_1 + q_2) = \xi q_1 + \xi q_2$,
3. $\xi(q_1 q_2) = (\xi q_1)q_2$, and
4. $\xi 1 = \xi$.

Definition 1.2.1. A Hermitian inner product on a right $\mathbb{H}-$module is a quaternionic-valued bilinear form on it, which is right-linear in the second slot, and is positive definite. That is, it satisfies the following properties.

1. $< \xi, \xi_1 + \xi_2 >=< \xi, \xi_1 > + < \xi, \xi_2 >$,
2. $< \xi_1, \xi_2 q >=< \xi_1, \xi_2 > q$,
3. $\overline{< \xi_2, \xi_1 >} =< \xi_1, \xi_2 >$, and
4. $< \xi, \xi >\geq 0$ and $< \xi, \xi >= 0$ if and only if $\xi = 0$.

The basic example is the form

$$< \xi, \eta >= \xi^* \eta = \sum_{i=1}^{n} \overline{\xi_i} \eta_i, \tag{1.2.1}$$

(by considering ξ as column matrix) on the right $\mathbb{H}-$module $\mathbb{H}^n$.

Using (2) and (3) of the definition above, we find that

$$< \xi_1 q, \xi_2 >= \overline{q} < \xi_1, \xi_2 > .$$

Indeed

$$\begin{aligned} \langle \xi_1 q, \xi_2 \rangle &= \overline{\langle \xi_2, \xi_1 q \rangle} = \overline{\langle \xi_2, \xi_1 \rangle q} \\ &= \overline{q}\overline{\langle \xi_2, \xi_1 \rangle} = \overline{q}\langle \xi_1, \xi_2 \rangle. \end{aligned}$$

On the complex vector space $\mathbb{C}^{2n}$, we can also define an Hermitian inner product as in (1.2.1). The following proposition relates the Hermitian inner product on quaternionic vectors as above with that on the corresponding complex vectors defined by the map ρ. Notice that we will use the same notation for both inner products.

Lemma 1.2.1. If $\xi, \eta \in \mathbb{H}^n$, then

$$\langle \xi, \eta \rangle = \langle \rho(\xi), \rho(\eta) \rangle - \langle \rho(\xi), \hat{J}\rho(\eta) \rangle$$

Proof. Let ξ and η be the vectors $\xi = \xi_1 + \xi_2 j$ and $\eta = \eta_1 + \eta_2 j$ where $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathbb{C}$. Then we see that

$$\begin{aligned} \langle \xi, \eta \rangle &= \xi^* \eta = (\overline{\xi_1}^t - \xi_2^t j)(\eta_1 + \eta_2 j) \\ &= \overline{\xi_1}^t \eta_1 + \xi_2^t \overline{\eta_2} + (\overline{\xi_1}^t \eta_2 - \xi_2^t \overline{\eta_1}) j \\ &= \langle \rho(\xi), \rho(\eta) \rangle - \langle \rho(\xi), J\overline{\rho(\eta)} \rangle j \\ &= \langle \rho(\xi), \rho(\eta) \rangle - \langle \rho(\xi), \hat{J} \rho(\eta) \rangle. \end{aligned}$$

■

1.3 Symplectic group

We will denote the set of invertible $n \times n$ matrices over $\mathbb{H}$ by $GL(n, \mathbb{H})$. Now we define the symplectic Lie group over quaternions as

$$Sp(n) = \{Q \in Gl(n, \mathbb{H}) \mid < Q\xi, Q\eta >=< \xi, \eta > \quad \text{for all} \quad \xi, \eta \in \mathbb{H}^n\}.$$

Observe that, by definition,

$$< Q\xi, Q\eta >= (Q\xi)^* Q\eta = (\xi^* Q^*)(Q\eta) = \xi^*(Q^* Q\eta) =< \xi, Q^* Q\eta > .$$

Hence $Q \in Sp(n)$ if and only if $Q^*Q = I$. Thus by Proposition 1.1.3, we have $QQ^* = I$ as well. Such s matrix is called a *unitary* quaternionic matrix.

Proposition 1.3.1. $\Phi(Sp(n)) = \{N \in M(2n, \mathbb{C}) \mid N^*N = I \quad \text{and} \quad \hat{J}N = N\hat{J}\}$.

Proof. Observe that Φ is $*$-homomorphism and use Proposition 1.1.4. ■

Proposition 1.3.2. $\Phi(Sp(n))$ consist of $N \in M(2n, \mathbb{C})$ such that

$$\langle N\rho(\xi), N\rho(\eta) \rangle = \langle \xi, \eta \rangle, \quad \langle N\rho(\xi), \hat{J}N\rho(\eta) \rangle = \langle \xi, \hat{J}\eta \rangle, \tag{1.3.1}$$

for all $\xi, \eta \in \mathbb{H}^n$. In other word, two inner products $\langle \xi, \eta \rangle$ and $\langle \xi, J\overline{\eta} \rangle$ will be left unchanged if we substitute ξ, η with $N\rho(\xi), N\rho(\eta)$, respectively.

Proof. Let $Q \in M(n, \mathbb{H})$ and $\xi, \eta \in \mathbb{H}^n$. Then applying Lemmas 1.1.5 and 1.2.1, we obtain

$$\langle \xi, \eta \rangle = \langle \rho(\xi), \rho(\eta) \rangle - \langle \rho(\xi), \hat{J}\rho(\eta) \rangle$$

and

$$\langle Q\xi, Q\eta \rangle = \langle \Phi(Q)\rho(\xi), \Phi(Q)\rho(\eta) \rangle - \langle \Phi(Q)\rho(\xi), \hat{J}\Phi(Q)\, \rho(\eta) \rangle.$$

Now it is clear that if $Q \in Sp(n)$ then $\Phi(Q) \in M(2n, \mathbb{C})$ satisfies (1.3.1). On the other hand, assume that $N \in M(2n, \mathbb{C})$ satisfies in (1.3.1), then from the first equation we obtain that $N^*N = I$ and from the second one, $N^*\hat{J}N = \hat{J}$. This is nothing but $\hat{J}N = N\hat{J}$. Now the proof of Proposition 1.3.2 is done using Proposition 1.3.1. ■

See also [11], p.16-24.

The Lie algebra of symplectic group $Sp(n)$ is

$$\mathfrak{sp}(n) = \{A \in M(n, \mathbb{H}) \mid A^* + A = 0\}.$$

A Lie bracket is an anti-commutative bracket $[Q, P] = QP - PQ$. The dimension of $\mathfrak{sp}(n)$ as vector space over $\mathbb{R}$ is $2n^2 + n$. Notice that mentioned Lie group is compact, connected, semisimple, and simple connected Lie group. See the references [74, 11] for exact definitions.

1.4 Killing form in Symplectic Lie algebra

Now we compute the Killing form of the two general elements of symplectic group and then for two specific ones we see that Killing form can be expressed in term of the Hermitian inner product. To do so, let us for a while represent a quaternionic number q by $q = q^1 + q^i i + q^j j + q^k k$. The componentwise product of two quaternions p and q is defined by $< p, q >_r = p^1 q^1 + p^i q^i + p^j q^j + p^k q^k$ and so is the componentwise inner product of vectors ξ, η, by $< \xi, \eta >_r = \sum_{i=1}^n < \xi_i, \eta_i >_r$. It is easy to see that

$$< \xi, \eta >_r = \frac{1}{2}(< \xi, \eta > + < \eta, \xi >),$$

which shows that componentwise inner product is just symmetrization of the Hermitian inner product.

Lemma 1.4.1. The Killing form of two elements A and B of the symplectic Lie algebra is following:

$$K(A, B) = -4(n+1) \sum_{s=1}^{n} < A_{ss}, B_{ss} >_r -8(n+1) \sum_{p<q} < A_{pq}, B_{pq} >_r . \qquad (1.4.1)$$

The proof can be found in Appendix C. Let us represent a general element A of the symplectic Lie algebra as

$$A = \begin{pmatrix} p & -\bar{\mathbf{p}}^t \\ \mathbf{p} & \mathbf{P} \end{pmatrix},$$

where p is a pure quaternionic number, $\mathbf{p}$ is quaternionic vector and $\mathbf{P} \in \mathfrak{sp}(n-1)$. We define a projection of $\mathfrak{sp}(n)$ by

$$\pi(A) = \begin{pmatrix} 0 & -\bar{\mathbf{0}}^t \\ \mathbf{0} & \mathbf{P} \end{pmatrix}.$$

This is an orthogonal projection with respect to the Killing form, that is,

$$K(\pi(A), (1-\pi)(B)) = 0.$$

Notice that neether π nor $1-\pi$ is a Lie algebra homomorphism. Indeed

$$\begin{aligned} &(1-\pi)([A,B]) - [(1-\pi)(A),(1-\pi)(B)] = \\ &= [A,B] - [\pi(A),\pi(B)] - [(1-\pi)(A),(1-\pi)(B)] \\ &= [A,\pi(B)] + [\pi(A),B] \\ &= 2[\pi(A),\pi(B)]. \end{aligned}$$

Lemma 1.4.2.

$$K(A,B) = K(\pi(A),\pi(B)) + K((1-\pi)(A),(1-\pi)(B)).$$

Concerning the Killing form and Lie bracket of matrices of the type in the last lemma, we can easily prove the following lemma.

Lemma 1.4.3. For pure quaternionic numbers p, q, r, vectors $\mathbf{p}, \mathbf{q}$ and matrices $\mathbf{P}, \mathbf{Q}$ and $\mathbf{R}$ of dimension $n-1$ in $\mathfrak{sp}(n-1)$, we have that

$$\left[\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} C_p q & 0 \\ 0 & 0 \end{pmatrix}, \tag{1.4.2a}$$

$$\left[\begin{pmatrix} 0 & -\overline{\mathbf{p}}^T \\ \mathbf{p} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\overline{\mathbf{q}}^T \\ \mathbf{q} & 0 \end{pmatrix}\right] = \begin{pmatrix} C_{\mathbf{q}}\mathbf{p} & 0 \\ 0 & \mathbf{q}\overline{\mathbf{p}}^T - \mathbf{p}\overline{\mathbf{q}}^T \end{pmatrix}, \tag{1.4.2b}$$

$$\left[\begin{pmatrix} 0 & 0 \\ 0 & \mathbf{Q} \end{pmatrix}, \begin{pmatrix} 0 & -\overline{\mathbf{p}}^T \\ \mathbf{p} & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & -\overline{\mathbf{Q}\mathbf{p}}^T \\ \mathbf{Q}\mathbf{p} & 0 \end{pmatrix}, \tag{1.4.2c}$$

$$K\left(\left[\begin{pmatrix} 0 & -\overline{\mathbf{p}}^T \\ \mathbf{p} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\overline{\mathbf{q}}^T \\ \mathbf{q} & 0 \end{pmatrix}\right], \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}\right) = K\left(\begin{pmatrix} C_{\mathbf{q}}\mathbf{p} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}\right), \tag{1.4.2d}$$

$$K\left(\begin{pmatrix} p & 0 \\ 0 & \mathbf{P} \end{pmatrix}, \left[\begin{pmatrix} r & 0 \\ 0 & \mathbf{R} \end{pmatrix}, \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}\right]\right) = K\left(\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}, \left[\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}\right]\right). \tag{1.4.2e}$$

Proof. First and second equalities are trivial. For the second and third ones, we need just prove that

$$K\left(\begin{pmatrix} p & 0 \\ 0 & \mathbf{P} \end{pmatrix}, \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}\right) = K\left(\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}\right).$$

But this follows from

$$K\left(\begin{pmatrix} 0 & 0 \\ 0 & \mathbf{P} \end{pmatrix}, \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}\right) = 0,$$

by using the formula for the Killing form of two elements $\begin{pmatrix} 0 & 0 \\ 0 & \mathbf{P} \end{pmatrix}$ and $\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ as in the Lemma 1.4.1. ∎

Consequently, though the matrix $\left[\begin{pmatrix} 0 & -\overline{\mathbf{p}}^T \\ \mathbf{p} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\overline{\mathbf{q}}^T \\ \mathbf{q} & 0 \end{pmatrix}\right]$ is not of the type of one of the matrices inside the bracket, but the Killing form of that with the matrix of the type $\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$, would give rise to the Killing form of two matrices of former type.