Integrable Systems and Symplectic Geometry

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Chapter 2

Variational complex and geometric operators

2.1 Algebraic approach to the concept of geometric operator

Definition 2.1.1. Let $\mathfrak{A}$ be any Lie algebra over a ring containing the real numbers. Then we say that the real vector space M is left $\mathfrak{A}-$module if a bilinear operation is given which assign to each pair $\mathfrak{a} \in \mathfrak{A}, m \in M$ an element $\mathfrak{a} \cdot m \in M$ such that

$$\mathfrak{a}_1 \cdot \mathfrak{a}_2 \cdot m - \mathfrak{a}_2 \cdot \mathfrak{a}_1 \cdot m = [\mathfrak{a}_1, \mathfrak{a}_2] \cdot m.$$

This is called a *representation* of $\mathfrak{A}$ on M.

Example 2.1.1. Let X be a smooth finite-dimensional manifold. The elements of the Lie algebra $\mathfrak{A}$ are vector fields on X and M is $C^\infty(X, \mathbb{R})$. The operation $[\mathfrak{a}_1, \mathfrak{a}_2]$ is the commutator of vector fields and $\mathfrak{a} \cdot m$ is the result of the action of the vector field $\mathfrak{a}$ on the function $m \in M$. Notice that in this example, M has a ring structure. But this is not essential in the theory of Hamiltonian formalism, as in the next section which will be devoted to the variational calculus, we would have left $\mathfrak{A}-$module M, without having ring structure on it.

Our objective is to construct a Hamiltonian structure on the pair $(\mathfrak{A}, M)$ using only the structures present in $\mathfrak{A}$ and M. There are thus two basic operation: the commutator $[\mathfrak{a}_1, \mathfrak{a}_2]$ and the action $\mathfrak{a} \cdot m$. We shall attempt to express all operations in terms of these basic operations.

We first define forms and differentials of forms, analogous to what in the case of Example 2.1.1 is called the de Rham complex.

Definition 2.1.2. A q-form is a multilinear function ω on $\mathfrak{A}$ with values in M, that is $\omega : \mathfrak{A} \times \ldots \mathfrak{A} \to M$. A 0-form is by definition a fixed element $m \in M$. The differential or coboundary operator d^q is given by the formula

$$\begin{aligned} d^q\omega(\mathfrak{a}_1, \ldots, \mathfrak{a}_{q+1}) &= \sum_i (-1)^{i+1} \mathfrak{a}_i \cdot \omega(\mathfrak{a}_1, \ldots, \hat{\mathfrak{a}}_i, \ldots, \mathfrak{a}_{q+1}) \\ &\quad + \sum_{i<j} (-1)^i \omega(\mathfrak{a}_1, \ldots, \hat{\mathfrak{a}}_i, \ldots, [\mathfrak{a}_i, \mathfrak{a}_j], \ldots, \mathfrak{a}_{q+1}), \end{aligned} \tag{2.1.1}$$

where the notation $\hat{}$ means the argument under the hat sign will be removed.

Remark 2.1.1. In the de Rham complex one restricts attention to antisymmetric forms, and this has become the norm in Lie algebra cohomology. As Loday [46] pointed out in his study of Leibniz algebras, there is no need to do so. The only reason for the antisymmetry assumption is that if one works with Lie algebras on which a ring R acts by left multiplication (as in vector fields multiplied by functions) and where R itself is a nontrivial $\mathfrak{A}$-module, with rules like (cf. Palais [52])

$$[fx, y] = f[x, y] - y(f)x, \quad x, y \in \mathfrak{A}, f \in R,$$

then to show if ω is R-linear, which is the usual tensor assumption, one also has that $d^q\omega$ is R-linear, one needs to assume that ω is antisymmetric.

In particular, $d^0m : \mathfrak{A} \to M$ is a linear map with values in M; the value of the 1-form d^0m on an element $\mathfrak{a} \in \mathfrak{A}$ is given by the formula

$$d^0m(\mathfrak{a}) = \mathfrak{a} \cdot m.$$

If ξ is a 1-form, i.e., a linear mapping $\xi : \mathfrak{A} \to M$, then

$$d^1\xi(\mathfrak{a}_1, \mathfrak{a}_2) = \mathfrak{a}_1 \cdot \xi(\mathfrak{a}_2) - \mathfrak{a}_2 \cdot \xi(\mathfrak{a}_1) - \xi([\mathfrak{a}_1, \mathfrak{a}_2]).$$

We can simply see that $d^1d^0m = 0$ is equivalent to the principle axiom of $\mathfrak{A}-$module M, that is:

$$\begin{aligned} d^1d^0m(\mathfrak{a}_1, \mathfrak{a}_2) &= \mathfrak{a}_1 \cdot d^0m(\mathfrak{a}_2) - \mathfrak{a}_2 \cdot d^0m(\mathfrak{a}_1) - d^0m([\mathfrak{a}_1, \mathfrak{a}_2]) \\ &= \mathfrak{a}_1 \cdot \mathfrak{a}_2 \cdot m - \mathfrak{a}_2 \cdot \mathfrak{a}_1 \cdot m - [\mathfrak{a}_1, \mathfrak{a}_2] \cdot m = 0. \end{aligned}$$

One can verify that $d^{q+1}d^q = 0$ in general.

Definition 2.1.3. A q-form ω is called *closed* if $d^q\omega = 0$.

We denote by $C^q(\mathfrak{A}, M)$ the space of q-forms. We observe that $C^q(\mathfrak{A}, M)$ is also a left $\mathfrak{A}$-module; the action of $\mathfrak{a}$ is called the Lie derivative which is given by

$$L^0_{\mathfrak{a}}m = \mathfrak{a} \cdot m, \quad m \in C^0(\mathfrak{A}, M) = M, \tag{2.1.2}$$

$$(L^1_{\mathfrak{a}}\xi)(\mathfrak{b}) = \mathfrak{a} \cdot \xi(\mathfrak{b}) - \xi([\mathfrak{a}, \mathfrak{b}]), \quad \xi \in C^1(\mathfrak{A}, M). \tag{2.1.3}$$

The general formula is, with $q > 0$,

$$\iota^q_{\mathfrak{b}}L^q_{\mathfrak{a}} = L^{q-1}_{\mathfrak{a}}\iota^q_{\mathfrak{b}} - \iota^q_{[\mathfrak{a},\mathfrak{b}]},$$

where

$$\iota^q_{\mathfrak{a}}\omega(\mathfrak{b}_1, \dots, \mathfrak{b}_{q-1}) = \omega(\mathfrak{a}, \mathfrak{b}_1, \dots, \mathfrak{b}_{q-1}) \quad \text{for} \quad \omega \in C^q(\mathfrak{A}, M).$$

We can show that

$$L^q_{\mathfrak{a}_1}L^q_{\mathfrak{a}_2} - L^q_{\mathfrak{a}_1}L^q_{\mathfrak{a}_2} = L^q_{[\mathfrak{a}_1,\mathfrak{a}_2]}$$

and

$$L^q_{\mathfrak{a}} = \iota^{q+1}_{\mathfrak{a}} d^q + d^{q-1} \iota^q_{\mathfrak{a}}. \tag{2.1.4}$$

One can use these Cartan rules as axioms to derive formula (2.1.1).

Now we suppose that in the space of 1-forms, a subspace $\mathfrak{E}^* \subset C^1(\mathfrak{A}, M)$ is fixed which contains the differential of all 0-forms (i.e., of elements of M.) In concrete situations considered in the next section, $\mathfrak{E}^*$ will be specially described.

Let $H : \mathfrak{E}^* \to \mathfrak{A}$ be a linear operator.

Definition 2.1.4. The operator H is called *anti-symmetric if for any* $\xi_1, \xi_2 \in \mathfrak{E}^*$ *we have*

$$\xi_1(H\xi_2) = -\xi_2(H\xi_1).$$

Notation 2.1.1. Sometimes we use the notation $(\xi, \mathfrak{a})$ for $\xi(\mathfrak{a})$.

Definition 2.1.5. With any anti-symmetric operator $H : \mathfrak{E}^* \to \mathfrak{A}$, we connect a 2-form ω_H defined on the image $\mathrm{im}(H)$ by

$$\omega_H(\mathfrak{a}_1, \mathfrak{a}_2) = (H^{-1}\mathfrak{a}_2)(\mathfrak{a}_1), \quad \mathfrak{a}_1, \mathfrak{a}_2 \in \mathrm{im}(H), \tag{2.1.5}$$

where by $H^{-1}\mathfrak{a}_2$ we denote any element ξ_2 such that $H\xi_2 = \mathfrak{a}_2$.

Lemma 2.1.2. The bilinear map ω_H is indeed a well-defined 2-form, that is, independent of the choice of ξ_2 in Definition 2.1.5.

Proof. Suppose $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathrm{im}(H)$, i.e., they have the form $\mathfrak{a}_i = H\xi_i$. In addition assume that $\mathfrak{a}_2 = H\xi_3$. Since H is anti-symmetric, we see that

$$\xi_3(\mathfrak{a}_1) = \xi_3(H\xi_1) = -\xi_1(H\xi_3) = -\xi_1(\mathfrak{a}_2).$$

Likewise we have that $\xi_2(\mathfrak{a}_1) = -\xi_1(\mathfrak{a}_2)$. Hence there holds $\xi_3(\mathfrak{a}_1) = \xi_2(\mathfrak{a}_1)$ for any two members of $\xi_2, \xi_3 \in H^{-1}\mathfrak{a}_2$ and $\mathfrak{a}_1 \in \mathrm{im}(H)$. ■

Definition 2.1.6. We say that an anti-symmetric operator $H : \mathfrak{E}^* \to \mathfrak{A}$ is Hamiltonian if

1. The image of the operator H is a subalgebra of the Lie algebra $\mathfrak{A}$ and
2. $d^2\omega_H = 0$ on this subalgebra.

In order to indicate criteria that an operator be Hamiltonian, we must obviously reformulate the condition that the form ω_H defined by formula 2.1.5 be closed directly in terms of the form generating the operator H.

2.2 Formal Variational Calculus

This section is devoted to the background and set up for geometric operators such as Hamiltonian, symplectic operator. In particular we describe noncommutative variational calculus and explain how this can be set up in the framework of a matrix Lie algebra.

2.2.1 Classical formal variational calculus

In the formal variational calculus introduced by Gel'fand and Dikii, [28], [27], one starts with coordinates $u^\alpha, \alpha = 1, \dots, N$, which take their values in a field or division ring and then one defines A to be the algebra of sums and products in the u^α and its derivatives with respect to the independent variable x. In other words, A is a free associative algebra over the real numbers. One then defines $Der(A)$, the space of derivations on A as $\mathfrak{a} \in Der A$ satisfying

$$\mathfrak{a} ab = \mathfrak{a}(a)b + a\mathfrak{a}(b).$$

That means $\mathfrak{a}$ act on A according to the Leibniz rule.

Remark 2.2.1. If one takes A to be the space of continuous functions on the real line with pointwise multiplication, then $Der(A) = 0$.

The space $Der(A)$ is a Lie algebra with the bracket defined as

$$[\mathfrak{a}_1, \mathfrak{a}_2] = \mathfrak{a}_1\mathfrak{a}_2 - \mathfrak{a}_2\mathfrak{a}_1.$$

We can express $\mathfrak{a} \in Der(A)$ as

$$\mathfrak{a} = \sum_{\alpha=1}^{N}\sum_{k=0}^{\infty} a^\alpha_{(k)} \frac{\partial}{\partial u^\alpha_k}, \quad a^\alpha_{(k)} \in A.$$

Here the term $a^\alpha_{(k)} \dfrac{\partial}{\partial u^\alpha_k}$ stands for the replacement of u^α_k by $a^\alpha_{(k)}$, using the Leibniz rule.

Example 2.2.1. Let A be generated by $u_k, k = 0, \dots$. Then $\mathfrak{a} u_k u_l = a_{(k)} u_l + u_k a_{(l)}$.

Example 2.2.2. Let

$$D_x = \sum_{\alpha=1}^{N}\sum_{k=0}^{\infty} u^\alpha_{k+1} \frac{\partial}{\partial u^\alpha_k}.$$

Then $D_x \in Der(A)$.

Definition 2.2.1. Let

$$\mathfrak{E} = \{\mathfrak{a} \in Der(A) \,|\, [\mathfrak{a}, D_x] = 0\},$$

be the space of all derivations on A commuting with D_x. It is clear that $\mathfrak{E}$ is a Lie subalgebra of $Der(A)$ due to the Jacobi identity. Moreover, since every $\mathfrak{a} \in Der(A)$ can be determined by its action on u^α_k, hence every $\mathfrak{a} \in \mathfrak{E}$ can be determined by its action on only u^α. Indeed if $\mathfrak{a} \in \mathfrak{E}$, then

$$a^\alpha_{(k)} = \mathfrak{a} u^\alpha_k = \mathfrak{a} D^k_x u^\alpha = D^k_x \mathfrak{a} u^\alpha = D^k_x a^\alpha_{(0)} =: a^\alpha_k,$$

so that $\mathfrak{a} \in \mathfrak{E}$ can be identified with some $(a^1, \dots, \alpha^N) \in A^N$, and we can write $\mathfrak{a} \in \mathfrak{E}$ as

$$\mathfrak{a} = \sum_{\alpha=1}^{N}\sum_{i=0}^{\infty} a^\alpha_i \frac{\partial}{\partial u^\alpha_i} \quad ,$$

with the replacement rule as before. We call elements of the Lie subalgebra $\mathfrak{E}$ *evolutionary vector fields*. The map $\mathfrak{p} : A^N \to \mathfrak{E}$ assigning $(a^1, \dots, a^N)$ to $\mathfrak{a}$ is called the *prolongation map*.

Definition 2.2.2. Let W be an arbitrary module over $\mathfrak{E}$. Then the Fréchet derivative of $S \in W$ in the direction $\mathfrak{a} \in \mathfrak{E}$, is defined as

$$D_S[\mathfrak{a}] = \frac{d}{d\epsilon} S(u + \epsilon \mathfrak{a} u)|_{\epsilon=0} \in W$$

Example 2.2.3. One has $D_{D_x}[\mathfrak{a}] = 0$, since D_x is not u-dependent.

Lemma 2.2.2. The Lie bracket on $\mathfrak{E}$ induces a Lie bracket on A^N:

$$[a, b]_A = D_b[\mathfrak{p}(a)] - D_a[\mathfrak{p}(b)], \quad a, b \in A^N.$$

One has $[\mathfrak{p}(a), \mathfrak{p}(b)]_{\mathfrak{E}} = \mathfrak{p}([a, b]_A)$, that is, $\mathfrak{p}$ is a Lie algebra homomorphism.

Proof. Assume that $a, b \in A^N$ and $c \in A$ are given. Then

$$\begin{aligned}
& [\mathfrak{p}(a), \mathfrak{p}(b)]_{\mathfrak{E}}(c) = \mathfrak{p}(a)(\mathfrak{p}(b)c) - \mathfrak{p}(b)(\mathfrak{p}(a)c) \\
= & \sum_{\alpha=1,\beta=1}^{N} \sum_{i=0,j=0}^{\infty} a_j^\beta \frac{\partial}{\partial u_j^\beta}\Big(b_i^\alpha \frac{\partial c}{\partial u_i^\alpha}\Big) - \sum_{\alpha=1,\beta=1}^{N} \sum_{i=0,j=0}^{\infty} b_j^\beta \frac{\partial}{\partial u_j^\beta}\Big(a_i^\alpha \frac{\partial c}{\partial u_i^\alpha}\Big) \\
= & \sum_{\alpha=1,\beta=1}^{N} \sum_{i=0,j=0}^{\infty} \Big(a_j^\beta \frac{\partial b_i^\alpha}{\partial u_j^\beta}\Big)\Big(\frac{\partial c}{\partial u_i^\alpha}\Big) + \sum_{\alpha=1,\beta=1}^{N} \sum_{i=0,j=0}^{\infty} \Big(a_j^\beta b_i^\alpha\Big)\Big(\frac{\partial^2 c}{\partial u_j^\beta \partial u_i^\alpha}\Big) \\
& - \sum_{\alpha=1,\beta=1}^{N} \sum_{i=0,j=0}^{\infty} \Big(b_j^\beta \frac{\partial a_i^\alpha}{\partial u_j^\beta}\Big)\Big(\frac{\partial c}{\partial u_i^\alpha}\Big) - \sum_{\alpha=1,\beta=1}^{N} \sum_{i=0,j=0}^{\infty} \Big(b_j^\beta a_i^\alpha\Big)\Big(\frac{\partial^2 c}{\partial u_j^\beta \partial u_i^\alpha}\Big) \\
= & \sum_{\alpha=1,\beta=1}^{N} \sum_{i=0,j=0}^{\infty} \Big(a_j^\beta \frac{\partial b_i^\alpha}{\partial u_j^\beta}\Big)\Big(\frac{\partial c}{\partial u_i^\alpha}\Big) - \sum_{\alpha=1,\beta=1}^{N} \sum_{i=0,j=0}^{\infty} \Big(b_j^\beta \frac{\partial a_i^\alpha}{\partial u_j^\beta}\Big)\Big(\frac{\partial c}{\partial u_i^\alpha}\Big) \\
= & \mathfrak{p}(\mathfrak{p}(a)b)(c) - \mathfrak{p}(\mathfrak{p}(b)a)(c) \\
= & \mathfrak{p}(\mathfrak{p}(a)b - \mathfrak{p}(b)a)(c) \\
= & \mathfrak{p}([a, b]_A)(c).
\end{aligned}$$

Let us explain here some of terms we used. The term

$$\Big(a_j^\beta \frac{\partial b_i^\alpha}{\partial u_j^\beta}\Big)\Big(\frac{\partial c}{\partial u_i^\alpha}\Big),$$

is computed as follows. The monomial u_i^α in c will be replaced by the expression $a_j^\beta \dfrac{\partial b_i^\alpha}{\partial u_j^\beta}$, which is itself the result of replacing a_j^β with u_j^β in b_i^α according to the Leibniz rule in both process.

Also the term

$$\left(a_j^\beta b_i^\alpha\right)\left(\frac{\partial^2 c}{\partial u_j^\beta \partial u_i^\alpha}\right)$$

is understood similarly. That is just replacing u_i^α and u_j^β with b_i^α and a_j^β respectively in c.

Notice that $\mathfrak{p}(a)b = \left(\mathfrak{p}(a)b^1, \ldots, \mathfrak{p}(a)b^N\right) \in A^N$. Also we see that

$$\begin{aligned}
\mathfrak{p}(a)(c) &= \sum_{\alpha=1}^{N}\sum_{i=0}^{\infty} a_i^\alpha \frac{\partial c}{\partial u_i^\alpha} \\
&= \sum_{\alpha=1}^{N}\sum_{i=0}^{\infty} \frac{d}{d\epsilon} c[u_i^\alpha + \epsilon a_i^\alpha]|_{\epsilon=0} \\
&= \sum_{\alpha=1}^{N}\sum_{i=0}^{\infty} \frac{d}{d\epsilon} c[u_i^\alpha + \epsilon \mathfrak{p}(a)(u_i^\alpha)]|_{\epsilon=0} \\
&= \frac{d}{d\epsilon} c[u + \epsilon \mathfrak{p}(a)(u)]|_{\epsilon=0} \\
&= D_c[\mathfrak{p}(a)].
\end{aligned}$$

Hence $D_b[\mathfrak{p}(a)] = \mathfrak{p}(a)(b) \in A^N$. ■

Definition 2.2.3. Let V be a $\mathfrak{E}$-module. We define an equivalence relation on V by

$$a_1 \sim a_2 \quad \text{if and only if} \quad a_1 - a_2 = D_x.a_3 \quad \text{for} \quad a_1, a_2, a_3 \in V.$$

Notice that $D_x \in \mathfrak{E}$, so this is well defined. We denote equivalence classes by $\tilde{V} = V/D_xV$ and its elements by $\int a$ for $a \in V$; these are called *functionals*.

Definition 2.2.4. We define an action of $\mathfrak{E}$ on $\tilde{V}$ by $\mathfrak{a} \cdot \int a = \int \mathfrak{a} \cdot a$ for $\mathfrak{a} \in \mathfrak{E}, a \in V$.

The action is well defined: Let $b \in V$. Then we have that

$$\mathfrak{a} \cdot \int D_x.b = \int \mathfrak{a} \cdot (D_x.b) = \int [\mathfrak{a}, D_x].b + \int D_x.(\mathfrak{a} \cdot b) = 0.$$

So evolutionary vector fields and functionals go well together.

Lemma 2.2.3. The space $\tilde{V}$ is an $\mathfrak{E}-$module under the action just defined.

Proof. In fact

$$\begin{aligned}
\mathfrak{a}_1 \cdot \mathfrak{a}_2 \cdot \int a - \mathfrak{a}_2 \cdot \mathfrak{a}_1 \cdot \int a &= \mathfrak{a}_1 \cdot \int \mathfrak{a}_2 \cdot a - \mathfrak{a}_2 \cdot \int \mathfrak{a}_1 \cdot a \\
&= \int \mathfrak{a}_1 \cdot \mathfrak{a}_2 \cdot a - \mathfrak{a}_2 \cdot \mathfrak{a}_1 \cdot a \\
&= \int [\mathfrak{a}_1, \mathfrak{a}_2] \cdot a \\
&= [\mathfrak{a}_1, \mathfrak{a}_2] \cdot \int a.
\end{aligned}$$

■

In the case $V = A$, one has, due to the Leibniz rule, integration by part as $\int(D_x a)b = -\int a(D_x b)$. Now if $\mathfrak{a} = \mathfrak{p}(a), a \in A^N$, then for $b \in A$, we have that

$$\begin{aligned} \mathfrak{a} \cdot \int b &= \int \mathfrak{a} \cdot b \\ &= \sum_{\alpha=1}^{N} \int \sum_{i=0}^{\infty} a_i^\alpha \frac{\partial b}{\partial u_i^\alpha} \\ &= \sum_{\alpha=1}^{N} \int \sum_{i=0}^{\infty} (D_x^i a^\alpha) \frac{\partial b}{\partial u_i^\alpha} \\ &= \int \sum_{\alpha=1}^{N} a^\alpha \sum_{i=0}^{\infty} (-D_x)^i (\frac{\partial b}{\partial u_i^\alpha}) \end{aligned} \tag{2.2.1}$$

Definition 2.2.5. The partial variational derivation $\frac{\delta}{\delta u^\alpha} : A \to A$ is defined by

$$\frac{\delta}{\delta u^\alpha} = \sum_{i=0}^{\infty} (-D_x)^i \frac{\partial}{\partial u_i^\alpha}.$$

Lemma 2.2.4 (Gel'fand and Dikii[27])**.** We have that $\frac{\delta}{\delta u^\alpha} \circ D_x = 0$.

Proof. One has $[D_x, \dfrac{\partial}{\partial u_i^\alpha}] = \dfrac{\partial}{\partial u_{i-1}^\alpha}$. Therefore

$$\begin{aligned} \frac{\delta}{\delta u^\alpha} \circ D_x &= \sum_{i=0}^{\infty} (-D_x)^i \frac{\partial}{\partial u_i^\alpha} D_x \\ &= \sum_{i=0}^{\infty} (-D_x)^{i+1} \frac{\partial}{\partial u_i^\alpha} + \sum_{i=1}^{\infty} (-D_x)^i \frac{\partial}{\partial u_{i-1}^\alpha} \\ &= 0, \end{aligned}$$

and the result follows. ■

Thus one can define $\frac{\delta}{\delta u^\alpha}$ on $\tilde{A}$, that is, $\frac{\delta}{\delta u^\alpha} : \tilde{A} \to \tilde{A}$. This is usually called in literature the *gradient of functional*. The operator $\frac{\delta}{\delta u^\alpha}$ is called **Euler operator**.

According to (2.2.1) we can write the action of $\mathfrak{a}$ on $\tilde{A}$ in terms of variational derivatives:

$$\mathfrak{a} \cdot \int b = \sum_{\alpha=1}^{N} \int a^\alpha \frac{\delta b}{\delta u^\alpha}.$$

Remark 2.2.5. Some Hamiltonian operators associated with $(\mathfrak{a}, \tilde{A})$ have been shown to be connected with certain very interesting nonlinear partial differential equations (cf [28],[27] and [29]).

2.2.2 Variational calculus for geometric dynamical symbols

As we have seen in Chapter 1, we derived the operator acting on a certain subspace consisting of the functions with values in a Lie algebra $\mathfrak{g}$. For instance, this subspace in the symplectic geometry is the space of functions with values of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & p & -\bar{\mathbf{p}}^t \\ 0 & \mathbf{p} & 0 \end{pmatrix} \in \mathfrak{sp}_{n+2},$$

depending on the matrix

$$U = \omega(D_x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & u & -\bar{\mathbf{u}}^t \\ 0 & \mathbf{u} & 0 \end{pmatrix}$$

and its derivatives $U_i = D_x^i U$ for $i = 0, \ldots$.

Definition 2.2.6. Let $K \in C^2(\mathfrak{sp}_{n+2}, \mathbb{R})$. A space $\mathfrak{E}^* \subset C^1(\mathfrak{E}, \tilde{V})$ is given by the formula

$$\mathfrak{b}^\star(\mathfrak{a}) = \int K(\mathfrak{b}, \mathfrak{a}), \quad \mathfrak{b} \in \mathfrak{E}.$$

Here we assume that K is nondegenerate, that is, if $\mathfrak{b}^* = 0$ it follows that $\mathfrak{b} = 0$. We recall that here V is the associated algebra generated by single dynamical variables $u, \mathbf{u}$ and their derivatives and furthermore $\tilde{V}$ is defined as before.

Definition 2.2.7. The inner product on $\mathfrak{E}$ defined by

$$< \mathfrak{a}, \mathfrak{b} >= \int K(\mathfrak{a}, \mathfrak{b}),$$

induces an inner product on $\mathfrak{E}^*$ as follows:

$$< \mathfrak{a}^*, \mathfrak{b}^* >_* =< \mathfrak{a}, \mathfrak{b} > .$$

Definition 2.2.8. If we have nondegenerate pairing, then for the linear operator $H : \mathfrak{E}^* \to \mathfrak{E}$, we can define the adjoint operator $H^* : \mathfrak{E} \to \mathfrak{E}^*$ by

$$< \mathfrak{a}, H\mathfrak{b}^* >=< H^*\mathfrak{a}, \mathfrak{b}^* >_* .$$

Now if $H : \mathfrak{E}^* \to \mathfrak{E}$ is anti-symmetric, then by definition $\mathfrak{a}^*(H\mathfrak{b}^*) = -\mathfrak{b}^*(H\mathfrak{a}^*)$. Hence

$$\begin{aligned} - < H^*\mathfrak{b}, \mathfrak{a}^* >_* &= - < \mathfrak{b}, H\mathfrak{a}^* >= -\mathfrak{b}^*(H\mathfrak{a}^*) \\ &= \mathfrak{a}^*(H\mathfrak{b}^*) =< \mathfrak{a}, H\mathfrak{b}^* >=< \mathfrak{a}^*, (H\mathfrak{b}^*)^* >_* . \end{aligned}$$

Thus $(H\mathfrak{b}^*)^* = -H^*\mathfrak{b}$.

Remark 2.2.6. Following the definition of adjoint operator H^* of $H : \mathfrak{E}^* \to \mathfrak{E}$ we see that

$$\mathfrak{c}^*(H\mathfrak{b}^*) = (H^*\mathfrak{c})\mathfrak{b}. \tag{2.2.2}$$

Indeed

$$\begin{aligned} \mathfrak{c}^*(H\mathfrak{b}^*) =< \mathfrak{c}, H\mathfrak{b} > &= < H^*\mathfrak{c}, \mathfrak{b}^* >_* \\ &= < (H^*\mathfrak{c})^*, \mathfrak{b} >= ((H^*\mathfrak{c})^*)^*(\mathfrak{b}) = (H^*\mathfrak{c})\mathfrak{b} \end{aligned}$$

Lemma 2.2.7. We have that $D_{\mathfrak{a}^*}[\mathfrak{c}] = (D_a c)^*$. Moreover, given the anti-symmetric operator $H : \mathfrak{E}^* \to \mathfrak{E}$ we have that

$$D_{H^*}[\mathfrak{a}](\mathfrak{c}) = -(D_H[a](\mathfrak{c}^*))^*, \quad \text{where} \quad H^* : \mathfrak{E} \to \mathfrak{E}^*. \tag{2.2.3}$$

Proof.

$$\begin{aligned} D_{\mathfrak{a}^*}[\mathfrak{c}](\mathfrak{b}) &= \frac{d}{d\epsilon}\mathfrak{a}^*[u + \epsilon\mathfrak{c}](\mathfrak{b}) \\ &= \frac{d}{d\epsilon}\int K(\mathfrak{a}[u + \epsilon\mathfrak{c}], \mathfrak{b}) \\ &= \int K(\frac{d}{d\epsilon}\mathfrak{a}[u + \epsilon\mathfrak{c}], \mathfrak{b}) \\ &= \int K(D_a c, \mathfrak{b}) \\ &= (D_a c)^*(\mathfrak{b}). \end{aligned}$$

Now for the anti-symmetric operator $H : \mathfrak{E}^* \to \mathfrak{E}$ we have that

$$\begin{aligned} D_{H^*}[\mathfrak{a}](\mathfrak{c}) &= \Big(\frac{d}{d\epsilon}H^*[u + \epsilon\mathfrak{a}]\Big)(\mathfrak{c}) \\ &= \frac{d}{d\epsilon}\Big(H^*[u + \epsilon\mathfrak{a}](\mathfrak{c})\Big) \\ &= -\frac{d}{d\epsilon}\Big(H[u + \epsilon\mathfrak{a}](\mathfrak{c}^*)\Big)^* \\ &= -\Big(\frac{d}{d\epsilon}H[u + \epsilon\mathfrak{a}](\mathfrak{c}^*)\Big)^* \\ &= -\Big(D_H[\mathfrak{a}](\mathfrak{c}^*)\Big)^*. \end{aligned}$$

■

Lemma 2.2.8. We do have that

1. $D_H[\mathfrak{a}](\mathfrak{b}^*) = D_{H\mathfrak{b}^*}[\mathfrak{a}] - H(D_{\mathfrak{b}^*}[\mathfrak{a}])$,
2. $D_{H^*}[\mathfrak{a}](\mathfrak{b}) = D_{H^*\mathfrak{b}}[\mathfrak{a}] - H^*(D_{\mathfrak{b}}[\mathfrak{a}])$

3. $D_{\mathfrak{c}^*}[\mathfrak{a}](\mathfrak{b}) = D_{\mathfrak{c}^*(\mathfrak{b})}[\mathfrak{a}] - \mathfrak{c}^*(D_{\mathfrak{b}}[\mathfrak{a}])$.

4. $D_S[\mathfrak{a}](\mathfrak{a}) = D_{S\mathfrak{b}}[\mathfrak{a}] - S(D_{\mathfrak{b}}\mathfrak{a})$. For the operator $S : \mathfrak{E} \to \mathfrak{E}^*$.

Proof.

$$\begin{aligned} D_{\mathfrak{c}^*(\mathfrak{b})}[\mathfrak{a}] &= \frac{d}{d\epsilon}\mathfrak{c}^*(u+\epsilon\mathfrak{a}u)(\mathfrak{b}(u+\epsilon\mathfrak{a}u))|_{\epsilon=0} \\ &= D_{\mathfrak{c}^*}[\mathfrak{a}](\mathfrak{b}) + \mathfrak{c}^*(D_{\mathfrak{b}}[\mathfrak{a}]) \end{aligned}$$

■

Lemma 2.2.9. For the operator $H : \mathfrak{E}^* \to \mathfrak{E}$, the following identity holds:

$$\mathfrak{c}^*\Big(D_H[\mathfrak{a}]\mathfrak{b}^*\Big) = \mathfrak{b}^*\Big((D_{H^*}[\mathfrak{a}](\mathfrak{c}))^*\Big).$$

Moreover if H is anti-symmetric we have that

$$\mathfrak{c}^*\Big(D_H[\mathfrak{a}]\mathfrak{b}^*\Big) = -\mathfrak{b}^*(D_H[\mathfrak{a}](\mathfrak{c}^*)).$$

Proof. Firstly we have that

$$\begin{aligned} D_{\mathfrak{c}^*(H\mathfrak{b}^*)}[\mathfrak{a}] &= \mathfrak{c}^*(D_{H\mathfrak{b}^*}[\mathfrak{a}]) + D_{\mathfrak{c}^*}[\mathfrak{a}](H\mathfrak{b}^*) \\ &= \mathfrak{c}^*(H(D_{\mathfrak{b}^*}[\mathfrak{a}])) + \mathfrak{c}^*(D_H[\mathfrak{a}](\mathfrak{b}^*)) + D_{\mathfrak{c}^*}[\mathfrak{a}](H\mathfrak{b}^*) \end{aligned} \tag{2.2.4}$$

Similarly we get that

$$\begin{aligned} D_{(H^*\mathfrak{c})\mathfrak{b}}[\mathfrak{a}] &= (H^*\mathfrak{c})(D_{\mathfrak{b}}[\mathfrak{a}]) + D_{H^*\mathfrak{c}}[\mathfrak{a}](\mathfrak{b}) \\ &= (H^*\mathfrak{c})(D_{\mathfrak{b}}[\mathfrak{a}]) + H^*(D_{\mathfrak{c}}[\mathfrak{a}])(\mathfrak{b}) + (D_{H^*}[\mathfrak{a}](\mathfrak{c}))(\mathfrak{b}) \end{aligned} \tag{2.2.5}$$

Applying (2.2.2) we see that

$$\mathfrak{c}^*(H\mathfrak{b}^*) = (H^*\mathfrak{c})\mathfrak{b}, \quad D_{\mathfrak{c}^*}[\mathfrak{a}](H\mathfrak{b}^*) = (D_{\mathfrak{c}}[\mathfrak{a}])^*(H\mathfrak{b}^*) = (H^*D_{\mathfrak{c}}[\mathfrak{a}])\mathfrak{b},$$

and

$$\mathfrak{c}^*(H(D_{\mathfrak{b}^*}[\mathfrak{a}])) = \mathfrak{c}^*(H(D_{\mathfrak{b}}[\mathfrak{a}])^*) = (H^*\mathfrak{c})D_{\mathfrak{b}}[\mathfrak{a}].$$

Hence the two equations (2.2.4) and (2.2.5) yield the following identity.

$$\mathfrak{c}^*(D_H[\mathfrak{a}](\mathfrak{b}^*)) = (D_{H^*}[\mathfrak{a}](\mathfrak{c}))(\mathfrak{b}) = \mathfrak{b}^*\Big((D_{H^*}[\mathfrak{a}](\mathfrak{c}))^*\Big).$$

Now if H is anti-symmetric, then $(D_{H^*}[\mathfrak{a}](\mathfrak{c}))^* = -D_H[\mathfrak{a}](\mathfrak{c}^*)$ by Lemma 2.2.7. ■

Remark 2.2.10. Likewise we can define the adjoint of differential operator $D_{\mathfrak{a}}$ where $\mathfrak{a} \in \mathfrak{E}$ as an operator $D^*_{\mathfrak{a}} : \mathfrak{E} \to \mathfrak{E}$. Indeed we define it as

$$< D^*_{\mathfrak{a}}\mathfrak{b}, \mathfrak{c} > = < \mathfrak{b}, D_{\mathfrak{a}}\mathfrak{c} > .$$

Remark 2.2.11. Assuming that K is nondegenerate, then we can identify $\mathfrak{E}^*$ with $\mathfrak{E}$. Hence if $H : \mathfrak{E}^* \to \mathfrak{E}$ is anti-symmetric, then the map $H : \mathfrak{E} \to \mathfrak{E}$ satisfies $H^* = -H$.

Lemma 2.2.12. Let $m = \int K(\mathfrak{b}, \mathfrak{c})$ and $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathfrak{E}$. Then

$$\mathfrak{a} \cdot m = \int K(\mathfrak{a}, D^*_{\mathfrak{b}}\mathfrak{c} + D^*_{\mathfrak{c}}\mathfrak{b}).$$

Proof. We compute

$$\begin{aligned}
\mathfrak{a} \cdot m &= \int K(D_{\mathfrak{a}}\mathfrak{b}, \mathfrak{c}) + K(\mathfrak{b}, D_{\mathfrak{a}}\mathfrak{c}) \\
&= \int K(D_{\mathfrak{b}}\mathfrak{a}, \mathfrak{c}) + K(\mathfrak{b}, D_{\mathfrak{c}}\mathfrak{a}) + \int K([\mathfrak{a}, \mathfrak{b}], \mathfrak{c}) + K(\mathfrak{b}, [\mathfrak{a}, \mathfrak{c}]) \\
&= \int K(\mathfrak{a}, D^\star_{\mathfrak{b}}\mathfrak{c}) + K(D^\star_{\mathfrak{c}}\mathfrak{b}, \mathfrak{a}) \\
&= \int K(\mathfrak{a}, D^\star_{\mathfrak{b}}\mathfrak{c} + D^\star_{\mathfrak{c}}\mathfrak{b}),
\end{aligned}$$

where we used the invariance of the Killing form. ■

Corollary 2.2.13. $d^0 m \in \mathfrak{E}^*$ for each $m \in \tilde{A}$.

We can define the Lie derivative of the operators between $\mathfrak{E}$ and $\mathfrak{E}^*$ similarly to that of the q-forms. For instance the Lie derivative of $H : \mathfrak{E}^* \to \mathfrak{E}$ is the map $L_{\mathfrak{a}}H : \mathfrak{E}^* \to \mathfrak{E}$ defined by $L_{\mathfrak{a}}H(\mathfrak{b}^*) = L_{\mathfrak{a}}(H\mathfrak{b}^*) - H(L_{\mathfrak{a}}\mathfrak{b}^*)$. Using the last lemma, we can write the Lie derivative of all differential object in terms of the Fréchet derivatives. In order to do this we need the following definition.

Definition 2.2.9. Given the operator $H : \mathfrak{E} \to \mathfrak{F}$, the *conjugate operator* $H^\dagger : \mathfrak{F}^* \to \mathfrak{E}^*$ is defined as

$$(H^\dagger(\mathfrak{f}^*))(\mathfrak{e}) = \mathfrak{f}^*(H\mathfrak{e}).$$

Lemma 2.2.14. The Lie derivative of the basic objects with respect to $\mathfrak{b} \in \mathfrak{E}$ are given by the following formulas:

$$\mathfrak{a} \in \mathfrak{E}, \qquad L_{\mathfrak{b}}\mathfrak{a} = D_{\mathfrak{a}}\mathfrak{b} - D_{\mathfrak{b}}\mathfrak{a}, \tag{2.2.6a}$$
$$\xi \in \mathfrak{E}^*, \qquad L_{\mathfrak{b}}\xi = D_\xi h + D^\dagger_{\mathfrak{b}}\xi. \tag{2.2.6b}$$
$$H : \mathfrak{E}^* \to \mathfrak{E}, \qquad L_{\mathfrak{b}}H = D_H\mathfrak{b} - D_{\mathfrak{b}}H - HD^\dagger_{\mathfrak{b}}, \tag{2.2.6c}$$
$$I : \mathfrak{E} \to \mathfrak{E}^*, \qquad L_{\mathfrak{b}}I = D_I\mathfrak{b} + ID_{\mathfrak{b}} + D^\dagger_{\mathfrak{b}}I, \tag{2.2.6d}$$
$$S : \mathfrak{E} \to \mathfrak{E}, \qquad L_{\mathfrak{b}}S = D_S\mathfrak{b} - D_{\mathfrak{b}}S + SD_{\mathfrak{b}}, \tag{2.2.6e}$$
$$T : \mathfrak{E}^* :\to \mathfrak{E}^*, \qquad L_{\mathfrak{b}}T = D_T\mathfrak{b} + D^\dagger_{\mathfrak{b}}T - TD^\dagger_{\mathfrak{b}}, \tag{2.2.6f}$$

Proof. 1. The first one is trivial.

2. For the second one, indeed we have

$$\begin{aligned} L_{\mathfrak{b}}\xi(\mathfrak{a}) &= L_{\mathfrak{b}}(\xi(\mathfrak{a})) - \xi(L_{\mathfrak{b}}\mathfrak{a}) \\ &= (D_\xi \mathfrak{b})(\mathfrak{a}) + \xi(D_{\mathfrak{a}}\mathfrak{b}) - \xi(D_{\mathfrak{a}}\mathfrak{b} - D_{\mathfrak{b}}\mathfrak{a}) \\ &= (D_\xi \mathfrak{b})(\mathfrak{a}) + \xi(D_{\mathfrak{b}}\mathfrak{a}) \\ &= (D_\xi \mathfrak{b})(\mathfrak{a}) + (D^\dagger_{\mathfrak{b}}\xi)(\mathfrak{a}) \end{aligned}$$

Hence we have that

$$L_{\mathfrak{b}}\xi = D_\xi \mathfrak{b} + D^\dagger_{\mathfrak{b}}\xi.$$

3. To prove third one we compute

$$\begin{aligned} L_{\mathfrak{b}}H(\xi) &= L_{\mathfrak{b}}(H\xi) - H(L_{\mathfrak{b}}\xi) \\ &= D_{H\xi}\mathfrak{b} - D_{\mathfrak{b}}H\xi - H(D_\xi \mathfrak{b} + D^\dagger_{\mathfrak{b}}\xi) \\ &= D_H\mathfrak{b}(\xi) + H(D_\xi \mathfrak{b}) - D_{\mathfrak{b}}H\xi - H(D_\xi \mathfrak{b} + D^\dagger_{\mathfrak{b}}\xi) \\ &= D_H\mathfrak{b}(\xi) - D_{\mathfrak{b}}H\xi - H(D^\dagger_{\mathfrak{b}}\xi) \end{aligned}$$

Hence we have that $L_{\mathfrak{b}}H(\xi) = D_H\mathfrak{b} - D_{\mathfrak{b}}H - HD^\dagger_{\mathfrak{b}}$.

4.

$$\begin{aligned} L_{\mathfrak{b}}I(\mathfrak{a}) &= L_{\mathfrak{b}}(I\mathfrak{a}) - I(L_{\mathfrak{b}}\mathfrak{a}) \\ &= D_{I\mathfrak{a}}\mathfrak{b} + D^\dagger_{\mathfrak{b}}I\mathfrak{a} - I(D_{\mathfrak{a}}\mathfrak{b} - D_{\mathfrak{b}}\mathfrak{a}) \\ &= D_I\mathfrak{b}(\mathfrak{a}) + I(D_{\mathfrak{a}}\mathfrak{b}) + D^\dagger_{\mathfrak{b}}I\mathfrak{a} - I(D_{\mathfrak{a}}\mathfrak{b} - D_{\mathfrak{b}}\mathfrak{a}) \\ &= D_I\mathfrak{b}(\mathfrak{a}) + D^\dagger_{\mathfrak{b}}I\mathfrak{a} - I(D_{\mathfrak{b}}\mathfrak{a}) \end{aligned}$$

Hence $L_{\mathfrak{b}}I = D_I\mathfrak{b} + D^\dagger_{\mathfrak{b}}I + ID_{\mathfrak{b}}$.

5.

$$\begin{aligned} (L_{\mathfrak{b}}S)(\mathfrak{a}) &= L_{\mathfrak{b}}(S\mathfrak{a}) - S(L_{\mathfrak{b}}\mathfrak{a}) \\ &= D_{S\mathfrak{a}}\mathfrak{b} - D_{\mathfrak{b}}S\mathfrak{a} - S(D_{\mathfrak{a}}\mathfrak{b} - D_{\mathfrak{b}}\mathfrak{a}) \\ &= D_S\mathfrak{b}(\mathfrak{a}) + S(D_{\mathfrak{a}}\mathfrak{b}) - D_{\mathfrak{b}}S\mathfrak{a} - S(D_{\mathfrak{a}}\mathfrak{b} - D_{\mathfrak{b}}\mathfrak{a}) \\ &= D_S\mathfrak{b}(\mathfrak{a}) - D_{\mathfrak{b}}S\mathfrak{a} + S(D_{\mathfrak{b}}\mathfrak{a}) \end{aligned}$$

Thus $L_{\mathfrak{b}}S = D_S\mathfrak{b} - D_{\mathfrak{b}}S + SD_{\mathfrak{b}}$.

6.

$$\begin{aligned} L_{\mathfrak{b}}T(\xi) &= L_{\mathfrak{b}}(T\xi) - T(L_{\mathfrak{b}}\xi) \\ &= D_{T\xi}\mathfrak{b} + D^\dagger_{\mathfrak{b}}T\xi - T(D_\xi \mathfrak{b} + D^\dagger_{\mathfrak{b}}\xi) \\ &= D_T\mathfrak{b}(\xi) + T(D_\xi \mathfrak{b}) + D^\dagger_{\mathfrak{b}}T\xi - T(D_\xi \mathfrak{b} + D^\dagger_{\mathfrak{b}}\xi) \\ &= D_T\mathfrak{b}(\xi) + D^\dagger_{\mathfrak{b}}T\xi - T(D^\dagger_{\mathfrak{b}}\xi) \end{aligned}$$

Therefore $L_{\mathfrak{b}}T = D_T\mathfrak{b} + D^\dagger_{\mathfrak{b}}T - TD^\dagger_{\mathfrak{b}}$.

■

2.3 Schouten bracket

In this section an effective algebraic characterization of a Hamiltonian operator will be given.

Definition 2.3.1. The Schouten bracket $[H,H]$ for the operator $H:\mathfrak{E}^*\to\mathfrak{E}$ is defined as

$$[H,H](\mathfrak{a}^*,\mathfrak{b}^*,\mathfrak{c}^*)=(L_{H\mathfrak{a}^*}\mathfrak{b}^*)(H\mathfrak{c}^*)+cycl.,$$

for $\mathfrak{a}^*,\mathfrak{b}^*,\mathfrak{c}^*\in\mathfrak{E}^*$.

Notice that the Schouten bracket can be expressed in terms of Fréchet derivatives:

Proposition 2.3.1. For the antisymmetric operator $H:\mathfrak{E}^*\to\mathfrak{E}$ we have that

$$\begin{aligned}&[H,H](\mathfrak{a}^*,\mathfrak{b}^*,\mathfrak{c}^*)\\ =\;&\mathfrak{b}^*(D_H[H\mathfrak{a}^*](\mathfrak{c}^*))+\mathfrak{c}^*(D_H[H\mathfrak{b}^*](\mathfrak{a}^*))+\mathfrak{a}^*(D_H[H\mathfrak{c}^*](\mathfrak{b}^*))\\ =\;&<D_H[H\mathfrak{a}^*](\mathfrak{c}^*),\mathfrak{b}>+<D_H[H\mathfrak{b}^*](\mathfrak{a}^*),\mathfrak{c}>+<D_H[H\mathfrak{c}^*](\mathfrak{b}^*),\mathfrak{a}>.\end{aligned}$$

Proof. First notice that for $\mathfrak{a}^*,\mathfrak{b}^*,\mathfrak{c}^*\in\mathfrak{E}^*$, by definition we have

$$\begin{aligned}&(L_{H\mathfrak{a}^*}\mathfrak{b}^*)(H\mathfrak{c}^*)=H\mathfrak{a}^*.\mathfrak{b}^*(H\mathfrak{c}^*)-\mathfrak{b}^*([H\mathfrak{a}^*,H\mathfrak{c}^*])\\ =\;&<H\mathfrak{a}^*,D^*_{\mathfrak{b}}H\mathfrak{c}^*+D^*_{H\mathfrak{c}^*}\mathfrak{b}>-<\mathfrak{b},D_{H\mathfrak{c}^*}H\mathfrak{a}^*-D_{H\mathfrak{a}^*}H\mathfrak{c}^*>\\ =\;&<D_{\mathfrak{b}}H\mathfrak{a}^*,H\mathfrak{c}^*>+<D_{H\mathfrak{c}^*}H\mathfrak{a}^*,\mathfrak{b}>-<\mathfrak{b},D_{H\mathfrak{c}^*}H\mathfrak{a}^*-D_{H\mathfrak{a}^*}H\mathfrak{c}^*>\\ =\;&<D_{\mathfrak{b}}H\mathfrak{a}^*,H\mathfrak{c}^*>+<\mathfrak{b},D_{H\mathfrak{a}^*}H\mathfrak{c}^*>\\ =\;&<D_{\mathfrak{b}}H\mathfrak{a}^*,H\mathfrak{c}^*>+<\mathfrak{b},D_H[H\mathfrak{c}^*](\mathfrak{a}^*)>+<\mathfrak{b},H(D_{\mathfrak{a}^*}[H\mathfrak{c}^*])>\\ =\;&<D_{\mathfrak{b}}H\mathfrak{a}^*,H\mathfrak{c}^*>+<\mathfrak{b},D_H[H\mathfrak{c}^*](\mathfrak{a}^*)>-<H\mathfrak{b}^*,D_{\mathfrak{a}}[H\mathfrak{c}^*]>.\end{aligned}$$

Now simply if we take cyclic permutation, then we get that

$$\begin{aligned}(L_{H\mathfrak{a}^*}\mathfrak{b}^*)(H\mathfrak{c}^*)+cycl.&=<\mathfrak{b},D_H[H\mathfrak{c}^*](\mathfrak{a}^*)>+cycl.\\ &=\mathfrak{b}^*(D_H[H\mathfrak{c}^*](\mathfrak{a}^*))+cycl.\\ &=[H,H](\mathfrak{a}^*,\mathfrak{b}^*,\mathfrak{c}^*),\end{aligned}$$

which proves the proposition. ∎

Definition 2.3.2. For operator $H:\mathfrak{E}^*\to\mathfrak{E}$ and $\mathfrak{b}^*\in\mathfrak{E}^*$ we define linear operator

$$D_H\mathfrak{b}^*:\mathfrak{E}\to\mathfrak{E},\quad\text{by}\quad D_H\mathfrak{b}^*(\mathfrak{a})=D_H[\mathfrak{a}](\mathfrak{b}^*).$$

The operator $D_H\mathfrak{b}^*$ is dual to the Fréchet derivative in some sense.

Lemma 2.3.2. For an antisymmetric operator $H:\mathfrak{E}^*\to\mathfrak{E}$, the Schouten bracket can be expressed as image of 1−form of its own argument.

$$[H,H](\mathfrak{a}^*,\mathfrak{b}^*,\mathfrak{c}^*)=\mathfrak{c}^*\Big(-(D_H\mathfrak{b}^*)(H\mathfrak{a}^*)+(D_H\mathfrak{a}^*)(H\mathfrak{b}^*)-H((D_H\mathfrak{b}^*)^\dagger\mathfrak{a}^*)\Big).$$

Proof. According to Definitions 2.2.9 and 2.3.2, we have that

$$\begin{aligned}
& [H,H](\mathfrak{a}^*,\mathfrak{b}^*,\mathfrak{c}^*) \\
= \; & \mathfrak{b}^*(D_H[H\mathfrak{a}^*](\mathfrak{c}^*)) + \mathfrak{c}^*(D_H[H\mathfrak{b}^*](\mathfrak{a}^*)) + \mathfrak{a}^*(D_H[H\mathfrak{c}^*](\mathfrak{b}^*)) \\
= \; & \mathfrak{b}^*(D_H[H\mathfrak{a}^*](\mathfrak{c}^*)) + \mathfrak{c}^*\Big((D_H\mathfrak{a}^*)(H\mathfrak{b}^*)\Big) + \mathfrak{a}^*\Big((D_H\mathfrak{b}^*)(H\mathfrak{c}^*)\Big) \\
= \; & \mathfrak{b}^*(D_H[H\mathfrak{a}^*](\mathfrak{c}^*)) + \mathfrak{c}^*\Big((D_H\mathfrak{a}^*)(H\mathfrak{b}^*)\Big) + \Big((D_H\mathfrak{b}^*)^\dagger\mathfrak{a}^*\Big)(H\mathfrak{c}^*).
\end{aligned}$$

Now applying Lemma 2.2.9, we obtain that

$$\begin{aligned}
& [H,H](\mathfrak{a}^*,\mathfrak{b}^*,\mathfrak{c}^*) \\
= \; & -\mathfrak{c}^*(D_H[H\mathfrak{a}^*](\mathfrak{b}^*)) + \mathfrak{c}^*\Big((D_H\mathfrak{a}^*)(H\mathfrak{b}^*)\Big) + \Big((D_H\mathfrak{b}^*)^\dagger\mathfrak{a}^*\Big)(H\mathfrak{c}^*).
\end{aligned}$$

From the fact that H is antisymmetric, the Schouten bracket will take following form:

$$\begin{aligned}
& [H,H](\mathfrak{a}^*,\mathfrak{b}^*,\mathfrak{c}^*) \\
= \; & -\mathfrak{c}^*((D_H\mathfrak{b}^*)(H\mathfrak{a}^*)) + \mathfrak{c}^*\Big((D_H\mathfrak{a}^*)(H\mathfrak{b}^*)\Big) - \mathfrak{c}^*(H((D_H\mathfrak{b}^*)^\dagger\mathfrak{a}^*)) \\
= \; & \mathfrak{c}^*\Big(-(D_H\mathfrak{b}^*)(H\mathfrak{a}^*) + (D_H\mathfrak{a}^*)(H\mathfrak{b}^*) - H((D_H\mathfrak{b}^*)^\dagger\mathfrak{a}^*)\Big),
\end{aligned}$$

which is the desired result. ■

In the following theorem, we give explicit criteria to decide whether an operator is Hamiltonian in terms of Fréchet derivatives.

Theorem 2.3.3. The anti-symmetric operator $H : \mathfrak{E}^* \to \mathfrak{E}$ is Hamiltonian according to Definition 2.1.6 if and only if $[H,H] = 0$.

Proof. First we prove that if $[H,H] = 0$ then $\mathrm{im}(H)$ is closed, or $\mathrm{im}(H)$ is a subalgebra of $\mathfrak{E}$. From the vanishing of the Schouten bracket, the last lemma and nondegeneracy, we have that

$$-(D_H\mathfrak{b}^*)(H\mathfrak{a}^*) + (D_H\mathfrak{a}^*)(H\mathfrak{b}^*) - H((D_H\mathfrak{b}^*)^\dagger\mathfrak{a}^*) = 0.$$

Then we can compute the bracket of two elements of $\mathrm{im}(H)$ as two elements of the Lie algebra $\mathfrak{E}$.

$$\begin{aligned}
& [H\mathfrak{a}^*, H\mathfrak{b}^*] \\
= \; & D_{H\mathfrak{b}^*}H\mathfrak{a}^* - D_{H\mathfrak{a}^*}H\mathfrak{b}^* \\
= \; & D_H[H\mathfrak{a}^*](\mathfrak{b}^*) + H(D_{\mathfrak{b}^*}[H\mathfrak{a}^*]) - D_H[H\mathfrak{b}^*](\mathfrak{a}^*) - H(D_{\mathfrak{a}^*}[H\mathfrak{b}^*]) \\
= \; & H\Big(D_{\mathfrak{b}^*}[H\mathfrak{a}^*] - D_{\mathfrak{a}^*}[H\mathfrak{b}^*]\Big) + D_H\mathfrak{b}^*(H\mathfrak{a}^*) - D_H\mathfrak{a}^*(H\mathfrak{b}^*) \\
= \; & H\Big(D_{\mathfrak{b}^*}[H\mathfrak{a}^*] - D_{\mathfrak{a}^*}[H\mathfrak{b}^*]\Big) - H((D_H\mathfrak{b}^*)^\dagger\mathfrak{a}^*) \\
= \; & H\Big(D_{\mathfrak{b}^*}[H\mathfrak{a}^*] - D_{\mathfrak{a}^*}[H\mathfrak{b}^*] - (D_H\mathfrak{b}^*)^\dagger\mathfrak{a}^*\Big).
\end{aligned}$$

Hence im(H) is subalgebra of the Lie algebra $\mathfrak{E}$. Thus the bilinear map ω_H is indeed a 2−form and we can compute its exterior derivative. Using the definition of action of $\mathfrak{E}$ on $\tilde{A}$, Lemmas 2.2.2 and 2.2.8, we can prove that (see Appendix A for the proof)

$$d^2\omega_H(\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3) = [H, H](\mathfrak{b}_1^*, \mathfrak{b}_2^*, \mathfrak{b}_3^*), \quad \mathfrak{a}_i = H\mathfrak{b}_i^*, i = 1, 2, 3.$$

Now the proof is complete. ■

Remark 2.3.4. In Chapter 4 we come up with the operator $H : \mathfrak{E} \to \mathfrak{E}$. In fact H is defined on a subspace of $\mathfrak{E}$, but simply we can just define the value of H on remaining space, which is nothing but $\mathfrak{sp}_n$, to be zero. This operator can be lifted to the operator $H : \mathfrak{E}^* \to \mathfrak{E}$, since we have identified $\mathfrak{E}^*$ with $\mathfrak{E}$. In the other words, we have taken a specific subspace of $C^1(\mathfrak{E}, \tilde{A})$. Hence we say that $H : \mathfrak{E} \to \mathfrak{E}$ is Hamiltonian if the related lifted operator $H : \mathfrak{E}^* \to \mathfrak{E}$ is Hamiltonian, that is the Schouten operator $[H, H]$ described in terms of Fréchet derivative as in Proposition 2.3.1 would vanish. We will discuss this in the next sections.

2.4 Symplectic operator

Let us now suppose that we have $\mathfrak{A}$−module M, and so we can build a complex (Ω, d) defined by the pair $(\mathfrak{A}, M)$. An specific example of which we will work later on would be the pair $(\mathfrak{E}, \tilde{A})$. In this section we will consider linear operators of the form $S : \mathfrak{E} \to \mathfrak{E}^*$ and define the symplectic operator by defining a symplectic structure as for Hamiltonian operator and then we will give a criteria in term of Fréchet derivatives for an operator to be symplectic in the formal variational complex.

Definition 2.4.1. The operator $S : \mathfrak{E} \to \mathfrak{E}^*$ is anti-symmetric if

$$(S\mathfrak{a})\mathfrak{b} = -(S\mathfrak{b})\mathfrak{a} \quad \text{for} \quad \mathfrak{a}, \mathfrak{b} \in \mathfrak{E}.$$

Let us assume, throughout this section, that we have the nondegenerate pairing between vector space $\mathfrak{E}^*$ and the Lie algebra $\mathfrak{E}$ as defined in the previous section.

Definition 2.4.2. The adjoint operator $S^* : \mathfrak{E} \to \mathfrak{E}^*$ for the operator $S : \mathfrak{E} \to \mathfrak{E}^*$ is defined due to the nondegeneracy of pairing as follows:

$$< S^*\mathfrak{b}^*, \mathfrak{a} >=< \mathfrak{b}^*, S\mathfrak{a} >_* .$$

If operator S is anti-symmetric, then $(S\mathfrak{a})^* = -S^*\mathfrak{a}^*$. This is easy to prove using non-degeneracy. Indeed

$$\begin{aligned} < S^*\mathfrak{b}^*, \mathfrak{a} >=< \mathfrak{b}^*, S\mathfrak{a} >_* &= < \mathfrak{b}, (S\mathfrak{a})^* >= (S\mathfrak{a})\mathfrak{b} \\ &= -(S\mathfrak{b})\mathfrak{a} =< -(S\mathfrak{b})^*, \mathfrak{a} > . \end{aligned}$$

Definition 2.4.3. Suppose that $S : \mathfrak{E} \to \mathfrak{E}^*$ is a linear operator. Then we define 2-form ω_S on $\mathfrak{E}$ by

$$\omega_S(\mathfrak{a}, \mathfrak{b}) = (S\mathfrak{b})(\mathfrak{a}). \tag{2.4.1}$$

It is clear that if S is anti-symmetric, so is ω_S.

Definition 2.4.4. The anti-symmetric linear operator $S : \mathfrak{E} \to \mathfrak{E}^*$ is called symplectic if the 2−form ω_S is closed.

Lemma 2.4.1. For an anti-symmetric operator $S : \mathfrak{E} \to \mathfrak{E}^*$, we have that

$$d^2\omega_S(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) = L_{\mathfrak{a}}(S\mathfrak{c})(\mathfrak{b}) + (cycl.) \tag{2.4.2}$$

where (cycl.) means terms with arguments cyclically permuted.

Proof.

$$\begin{aligned} d^2\omega_S(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) &= \mathfrak{a}.\omega_S(\mathfrak{b}, \mathfrak{c}) - \mathfrak{b}.\omega_S(\mathfrak{a}, \mathfrak{c}) + \mathfrak{c}.\omega_S(\mathfrak{a}, \mathfrak{b}) \\ &\quad -\omega_S([\mathfrak{a}, \mathfrak{b}], \mathfrak{c}) + \omega_S([\mathfrak{b}, \mathfrak{c}], \mathfrak{b}) - \omega_S([\mathfrak{b}, \mathfrak{c}], \mathfrak{a}) \\ &= \mathfrak{a}.S\mathfrak{c}(\mathfrak{b}) + \mathfrak{b}.S\mathfrak{a}(\mathfrak{c}) + \mathfrak{c}.S\mathfrak{b}(\mathfrak{a}) \\ &\quad -\omega_S([\mathfrak{a}, \mathfrak{b}], \mathfrak{c}) + \omega_S([\mathfrak{b}, \mathfrak{c}], \mathfrak{b}) - \omega_S([\mathfrak{b}, \mathfrak{c}], \mathfrak{a}) \\ &= L_{\mathfrak{a}}S\mathfrak{c}(\mathfrak{b}) + L_{\mathfrak{b}}S\mathfrak{a}(\mathfrak{c}) + L_{\mathfrak{c}}S\mathfrak{b}(\mathfrak{a}). \end{aligned}$$

■

As in the setup of the Hamiltonian operator, the right hand side of (2.4.2) can be formulated by defining a similar Schouten bracket $[S, S]$ for the operator $S : \mathfrak{E} \to \mathfrak{E}^*$. That is

$$[S, S](\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) = (L_{\mathfrak{a}}S\mathfrak{c})(\mathfrak{b}) + cycl.$$

Corollary 2.4.2. If the pairing is nondegenerate, then the anti-symmetric operator S is symplectic if and only if the Schouten bracket $[S, S](\mathfrak{a}, \mathfrak{b}, \mathfrak{c})$, vanishes for all $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ in $\mathfrak{E}$.

The Schouten bracket for the operator $S : \mathfrak{E} \to \mathfrak{E}^*$ can be expressed in terms of Fréchet derivatives.

Lemma 2.4.3. For an anti-symmetric operator $S : \mathfrak{E} \to \mathfrak{E}^*$, we have that

$$[S, S](\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) = L_{\mathfrak{a}}S\mathfrak{c}(\mathfrak{b}) + cycl. = -\int K\Big(D_{S^*}[\mathfrak{a}]\mathfrak{c}^*, \mathfrak{b}\Big) + cycl.$$

Proof. By Definition 2.1.3 we get that

$$\begin{aligned} L_{\mathfrak{a}}S\mathfrak{c}(\mathfrak{b}) &= \mathfrak{a}.(S\mathfrak{c}(\mathfrak{b})) - (S\mathfrak{c})([\mathfrak{a}, \mathfrak{b}]) = \mathfrak{a}. < (S\mathfrak{c})^*, \mathfrak{b} > -(S\mathfrak{c})([\mathfrak{a}, \mathfrak{b}]) \\ &= \int K(\mathfrak{a}, D^*_{(S\mathfrak{c})^*}\mathfrak{b} + D^*_{\mathfrak{b}}(S\mathfrak{c})^*) - \int K((S\mathfrak{c})^*, D_{\mathfrak{b}}\mathfrak{a} - D_{\mathfrak{a}}\mathfrak{b}) \\ &= \int K(D_{(S\mathfrak{c})^*}\mathfrak{a}, \mathfrak{b}) + \int K(D_{\mathfrak{b}}\mathfrak{a}, (S\mathfrak{c})^*) - \int K((S\mathfrak{c})^*, D_{\mathfrak{b}}\mathfrak{a} - D_{\mathfrak{a}}\mathfrak{b}) \\ &= \int K(D_{(S\mathfrak{c})^*}\mathfrak{a}, \mathfrak{b}) + \int K((S\mathfrak{c})^*, D_{\mathfrak{a}}\mathfrak{b}) \\ &= -\int K(D_{S^*\mathfrak{c}^*}\mathfrak{a}, \mathfrak{b}) - \int K(S^*\mathfrak{c}^*, D_{\mathfrak{a}}\mathfrak{b}) \\ &= -\int K\Big(D_{S^*}[\mathfrak{a}]\mathfrak{c}^* + S^*(D_{\mathfrak{c}^*}[\mathfrak{a}]), \mathfrak{b}\Big) - \int K(S^*\mathfrak{c}^*, D_{\mathfrak{a}}\mathfrak{b}) \\ &= -\int K\Big(D_{S^*}[\mathfrak{a}]\mathfrak{c}^*, \mathfrak{b}\Big) + \int K\Big(D_{\mathfrak{c}}\mathfrak{a}, S^*\mathfrak{b}^*\Big) - \int K(S^*\mathfrak{c}^*, D_{\mathfrak{a}}\mathfrak{b}) \end{aligned}$$

Hence by taking cyclic permutation, one can get rid of two last terms of last line, so that we get the equality as stated. ■

Remark 2.4.4. There is a similar observation as we explained in Remark 2.3.4. In fact to prove that the operator $S : \mathfrak{E} \to \mathfrak{E}$ is symplectic, we simply prove that the Schouten bracket $[S, S]$ described as in the last lemma vanishes using the properties of the Killing form and the Jacobi identity.

2.5 Nijenhuis operator

The concept of Nijenhuis operator has been introduced into the theory of integrable system in the work of Magri, Gelfand and Dorfman(see [14]) and under the name of hereditary operators, in that of Fuchssteiner and Fokas, see [24],[23]. The defining relation for this operator was originally found as a necessary condition for an almost complex structure to be complex, i.e., as an integrability condition. Its important property is to construct an abelian Lie algebras.

Definition 2.5.1. The linear map $N : \mathfrak{E} \to \mathfrak{E}$ is a *Nijenhuis operator* if

$$[N\mathfrak{x}, N\mathfrak{y}] - N[N\mathfrak{x}, \mathfrak{y}] - N[\mathfrak{x}, N\mathfrak{y}] + N^2[\mathfrak{x}, \mathfrak{y}] = 0 \tag{2.5.1}$$

for any pair $\mathfrak{x}, \mathfrak{y} \in \mathfrak{E}$.

In the local version, if u is dynamical symbol, then using the Fréchet derivatives, the condition (2.5.1) takes the form as in the following proposition.

Proposition 2.5.1. In local version, linear map $N : \mathfrak{E} \to \mathfrak{E}$ is Nijenhuis operator if and only if

$$D_N[N\mathfrak{y}](\mathfrak{x}) - D_N[N\mathfrak{x}](\mathfrak{y}) + N(D_N[\mathfrak{x}](\mathfrak{y}) - D_N[\mathfrak{y}](\mathfrak{x})) = 0. \tag{2.5.2}$$

for any pair $\mathfrak{x}, \mathfrak{y} \in \mathfrak{E}$.

Proof. For any pair $\mathfrak{x}, \mathfrak{y} \in \mathfrak{E}$. we have that

$$\begin{aligned}
& [N\mathfrak{x}, N\mathfrak{y}] - N[N\mathfrak{x}, \mathfrak{y}] - N[\mathfrak{x}, N\mathfrak{y}] + N^2[\mathfrak{x}, \mathfrak{y}] \\
= \; & D_{N\mathfrak{y}}N\mathfrak{x} - D_{N\mathfrak{x}}N\mathfrak{y} - N(D_{\mathfrak{y}}N\mathfrak{x} - D_{N\mathfrak{x}}\mathfrak{y}) - N(D_{N\mathfrak{y}}\mathfrak{x} - D_{\mathfrak{x}}N\mathfrak{y}) \\
& + N^2(D_{\mathfrak{y}}\mathfrak{x} - D_{\mathfrak{x}}\mathfrak{y}) \\
= \; & D_N[N\mathfrak{x}](\mathfrak{y}) + N(D_{\mathfrak{y}}N\mathfrak{x}) - D_N[N\mathfrak{y}](\mathfrak{x}) - N(D_{\mathfrak{x}}N\mathfrak{y}) \\
& - N(D_{\mathfrak{y}}N\mathfrak{x} - D_N[\mathfrak{y}](\mathfrak{x}) - N(D_{\mathfrak{x}}\mathfrak{y})) - N(D_N[\mathfrak{x}](\mathfrak{y}) + N(D_{\mathfrak{y}}\mathfrak{x}) - D_{\mathfrak{x}}N\mathfrak{y}) \\
& + N^2(D_{\mathfrak{y}}\mathfrak{x} - D_{\mathfrak{x}}\mathfrak{y}) \\
= \; & D_N[N\mathfrak{x}](\mathfrak{y}) - D_N[N\mathfrak{y}](\mathfrak{x}) + N(D_N[\mathfrak{y}](\mathfrak{x}) - D_N[\mathfrak{x}](\mathfrak{y})).
\end{aligned}$$

■

The following theorem explain how one can construct an abelian Lie algebra out of Nijenhuis operator. The proof can be found in [14], theorem 3.4.

Theorem 2.5.2. Let $\mathfrak{a} \in \mathfrak{E}$ be the symmetry of a Nijenhuis operator N, i.e., $L_{\mathfrak{a}} N = 0$. Then all elements of $N^j \mathfrak{a} \in \mathfrak{E}$ are symmetries of N for all $j, k > 0$. If N is invertible, the same is true for all $j, k \in \mathbb{Z}$. These symmetries commute.

An example of such an operator is given in Chapter 4. For more information, see [51]. Good applications are [49], [39] and [48].