Integrable Systems and Symplectic Geometry

Asadi, E.

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Chapter 3

Preliminaries of Geometry

3.1 Riemannian geometry

In this section we present all concepts about Riemannian manifolds that we need later on. A manifold will be $C^\infty-$manifold. A good reference for these matters would be [53].

Definition 3.1.1. Let U be a neighborhood of a point p on a C^∞-manifold M. We denote the ring of C^∞-functions on U by $C^\infty(U)$ and the space of derivations on $C^\infty(U)$ by $\mathfrak{X}^\infty(U)$.

Lemma 3.1.1. The space $\mathfrak{X}^\infty(U)$ is a Lie algebra.

Proof. Let $X_i, i = 1, 2$ be derivations, that is, $X_i(fg) = (X_i f)g + f(X_i g)$. The space $\mathfrak{X}^\infty(U)$ is an associative algebra, where the composition is given by $(X \cdot Y)(f) = X(Y(f))$. This implies antisymmetry and the Jacobi identity, when we define $[X,Y] = X \cdot Y - Y \cdot X$. One now has to show that the commutator is indeed a derivation. One has

$$
\begin{aligned}
[X,Y]fg &= (X \cdot Y - Y \cdot X)(fg) \\
&= X(Y(f)g + fY(g)) - Y(X(f)g + fX(g)) \\
&= XY(f)g + Y(f)X(g) + X(f)Y(g) + fXY(g) \\
&\quad -YX(f)g - X(f)Y(g) - Y(f)X(g) - fYX(g) \\
&= XY(f)g + fXY(g) - YX(f)g - fYX(g) \\
&= [X,Y](f)g + f[X,Y](g),
\end{aligned}
$$

and this implies that $[X,Y] \in \mathfrak{X}^\infty(U)$. ■

Definition 3.1.2. A *connection* on a manifold M is an operator $\nabla : \mathfrak{X}^\infty(U) \times \mathfrak{X}^\infty(U) \to \mathfrak{X}^\infty(U)$ which assigns to two C^∞ vector fields X and Y with domain U, a third C^∞ vector field denoted by $\nabla_X Y$ with the same domain U, in such a way that the following properties are satisfied:

1. $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$,
2. $\nabla_{X+W} Y = \nabla_X Y + \nabla_W Y$,
3. $\nabla_{fX} Y = f\nabla_X Y$,

4. $\nabla_X fY = X(f)Y + f\nabla_X Y$,

for any X, W vectors at $p \in \mathsf{M}, Y, Z$ smooth fields and f a smooth function defined on a neighborhood of p.

Definition 3.1.3. We say an $n-$dimensional manifold M is a *Riemannian manifold* if M is endowed with a symmetric and positive definite 2-covariant tensor field $<,>$, that is, it is $C^\infty(U)$-bilinear. The tensor $<,>$ is called the *Riemannian metric* of the manifold, and it allows us to define distances, length, angles, orthogonality, etc., in the natural way. In particular, the *length* of a vector X is defined as

$$\|X\| = \sqrt{\langle X, X\rangle}.$$

Definition 3.1.4. A *Riemannian connection* on a Riemannian manifold M is a connection ∇ on M such that

1. $\nabla_X Y - \nabla_Y X = [X, Y]$(the connection is *torsion free*),
2. $Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$,
   for all fields X, Y and Z with the common domain.

Definition 3.1.5. The *curvature tensor* of a connection ∇ is a tensor R that assigns to each pair of vectors X, Y at a point p a linear transformation $R(X, Y)$ of the tangent space to p, as $\mathsf{T}_p\mathsf{M}$, into itself. After extending X, Y and Z to smooth vector fields on U, $R(X, Y)Z$ is defined via the relation

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_X \nabla_Y Z - \nabla_{[X,Y]} Z. \tag{3.1.1}$$

That this defines a tensor has to be proved. The value of this expression is independent of the way the vector fields were extended.

Definition 3.1.6. The *torsion tensor* of a connection ∇ is defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

where X and Y are smooth vector fields on U.

Definition 3.1.7. The *Riemann-Christoffel* curvature tensor (of type $(0, 4)$) is the $4-$covariant tensor

$$K(X, Y, Z, W) = \langle R(Z, W)Y, X\rangle$$

for any X, Y, Z, and W tangent vectors at p.

Riemannian curvature tensors have the following properties:

Theorem 3.1.2. The following relations are true:

1. $K(X, Y, Z, W) = -K(Y, X, Z, W) = -K(X, Y, W, Z)$,

2. $K(X,Y,Z,W) = K(Z,W,X,Y)$.

Definition 3.1.8. Given two independent vectors $X, Y \in \mathsf{T}_p\mathsf{M}$, the normalized quadratic form,

$$sec(X,Y) = \frac{K(X,Y,X,Y)}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2},$$

is called *sectional curvature* of X, Y. It can easily be checked that $sec(X,Y)$ depends only on the plane π spanned by X and Y, and so the sectional curvature is also called $K(\pi)$, the Riemannian curvature of the plane section π.

Definition 3.1.9. A Riemannian manifold M is said to have *constant Riemannian curvature* $\varkappa$ if the Riemannian curvature of all plane sections is the constant $\varkappa$.

Proposition 3.1.3. The following properties are equivalent:

1. $K(\pi) = \varkappa$ for all 2-planes in $\mathsf{T}_p\mathsf{M}$.

2. $R(X,Y)Z = \varkappa(\langle Y,Z\rangle X - \langle X,Z\rangle Y)$ for any X, Y and Z in $\mathsf{T}_p\mathsf{M}$.

Corollary 3.1.4. Assume the manifold M has certain constant Riemannian curvature. Then

1. $\nabla_X R = 0$ along any direction determined by the vector field X. That is, the Riemannian curvature tensor is parallel.

2. If Z is orthogonal to X and Y, then $R(X,Y)Z = 0$.

3. If W is orthogonal to X and Y, then $K(W,Z,X,Y) = 0$ for any Z.

3.2 Cartan's moving frame method

This method was introduced by Elie Cartan at the beginning of last century. Cartan's insight was that the local properties of a manifold equipped with a geometric structure can be very well understood if one knows how the frames of the tangent bundle (compatible with the geometric structure) vary from one point of the manifold to another.

Let M be an arbitrary Riemannian manifold with metric $\langle , \rangle$. We choose a local orthonormal moving frame $X = \{\mathbf{e}_i \,|\, i = 1, \ldots, n\}$. Denote by (θ^i) the dual coframe, i.e.,

$$\theta^i(\mathbf{e}_j) = \delta^i_j.$$

Remark 3.2.1. Notice that the notion of "dual of frame" which is another frame is defined as another frame $Y = \{\mathbf{f}_i \,|\, i = 1, \ldots, n\}$ so that $\langle \mathbf{e}_i, \mathbf{f}_j\rangle = \delta_{ij}$. See [42] for more discussion on this issue.

Lemma 3.2.2 (E. Cartan)**.** On the Riemannian manifold M as above, there exists a collection of 1-forms ω^i_j uniquely defined by the requirements

(a) $\omega^i_j = -\omega^j_i$.

(b) $d\theta^i = \theta^j \wedge \omega^i_j$.

Proof. *Uniqueness*: Suppose ω^i_j satisfy the conditions mentioned above. Since

$$\{\theta^k \,|\, k = 1, \ldots, n\}$$

form a basic for 1-forms and

$$\{\theta^j \wedge \theta^k \,|\, j, k = 1, \ldots, n \quad \text{and} \quad j < k\}$$

a basic for 2-forms, there exist functions f^i_{jk} and g^i_{jk} such that

$$\omega^i_j = \sum_k f^i_{jk}\theta^k, \quad \text{and} \quad d\theta^i = \sum_{j<k} g^i_{jk}\theta^j \wedge \theta^k, \quad g^i_{jk} = -g^i_{kj},$$

so that we have then

$$\begin{aligned}
\sum_j^n \theta^j \wedge \omega^i_j &= \sum_{j,k=1}^n \theta^j \wedge f^i_{jk}\theta^k \\
&= -\sum_{j,k=1}^n f^i_{jk}\theta^k \wedge \theta^j \\
&= \sum_{j,k=1,j<k}^n (f^i_{jk} - f_{kj})\theta^j \wedge \theta^k
\end{aligned}$$

Then the condition (a) is equivalent to

(a1) $f^i_{jk} = -f^j_{ik}$
while (b) gives

(b1) $f^i_{jk} - f^i_{kj} = g^i_{jk}$.

The above two relations uniquely determine the f's in terms of the $g's$ via a cyclic permutation of the indices i, j, k as

$$f^i_{jk} = \frac{1}{2}(g^i_{jk} + g^j_{ki} - g^k_{ij}). \tag{3.2.1}$$

Existence: Consider the functions g^i_{jk} defined by

$$d\theta^i = \sum_{j<k} g^j_{jk}\theta^i \wedge \theta^k, \quad g^i_{jk} = -g^i_{kj}.$$

Next define $\omega^i_j = f^i_{jk}\theta^k$ where f's are given by (3.2.1). Then the forms ω^i_j satisfy both (a) and (b). ■

Let the matrix a be invertible, so that $X.a$ is another moving frame and the dual frame associated to this moving frame is simply

$$\theta_{X.a} = a^{-1}\theta_X,$$

in which θ_X is the dual 1-forms as above associated to X. In the following lemma we determine how 1-form ω behaves under the change of moving frame.

Lemma 3.2.3. We have that

$$\omega_{X.a} = a^{-1}da + a^{-1}\omega_X a, \tag{3.2.2}$$

where ω_X and $\omega_{X.a}$ are unique 1-forms as in Lemma 3.2.2 associated to moving frames X and $X.a$, respectively.

Proof. We compute

$$\begin{aligned} d(a\theta_{X.a}) &= da \wedge \theta_{X.a} + a.d\theta_{X.a} \\ &= da \wedge (a^{-1}.\theta_X) + a.(\theta_{X.a} \wedge \omega_{X.a}) \\ &= (da.a^{-1}) \wedge \theta_X - a.(\omega_{X.a} \wedge \theta_{X.a}) \\ &= (da.a^{-1}) \wedge \theta_X - a.(\omega_{X.a} \wedge a^{-1}\theta_X) \\ &= (da.a^{-1}) \wedge \theta_X - (a.\omega_{X.a}.a^{-1}) \wedge \theta_X \\ &= (da.a^{-1} - a.\omega_{X.a}.a^{-1}) \wedge \theta_X. \end{aligned}$$

On the other hand from the fact $a\theta_{X.a} = \theta_X$, we get that

$$d(a\theta_{X.a}) = d\theta_X = \theta_X \wedge \omega_X = -\omega_X \wedge \theta_X.$$

This implies that

$$da.a^{-1} - a.\omega_{X.a}.a^{-1} = -\omega_X, \quad \text{or} \quad \omega_{X.a} = a^{-1}.da + a^{-1}.\omega_X.a.$$

■

We see that the 1-form ω corresponding to the moving frame does not behave like a tensor when we change the coordinate system or moving frame.

Definition 3.2.1. *Cartan connection* on a manifold M is an assignment of a matrix valued 1-form ω to every moving frame such that (3.2.2) holds.

Lemma 3.2.4. Let ∇ be the Levi Civita connection on Riemannian manifold M compatible with its metric and $\mathbf{e}_i'$s and θ's are as mentioned above and also ω^i_j be the 1-forms as in Cartan's Lemma. Then

$$\nabla_{\mathbf{e}_k}\mathbf{e}_j = \omega^i_j(\mathbf{e}_k)\mathbf{e}_i.$$

Proof. Define $\hat{\omega}^i_j$ by

$$\nabla_{\mathbf{e}_k}\mathbf{e}_j = \hat{\omega}^i_j(\mathbf{e}_k)\mathbf{e}_i.$$

Since the connection is compatible, thus

$$\begin{aligned} 0 &= \nabla_{\mathbf{e}_k}\langle \mathbf{e}_j, \mathbf{e}_l\rangle &(3.2.3)\\ &= \langle \nabla_{\mathbf{e}_k}\mathbf{e}_j, \mathbf{e}_l\rangle + \langle \mathbf{e}_j, \nabla_{\mathbf{e}_k}\mathbf{e}_l\rangle &(3.2.4)\\ &= \langle \hat{\omega}^i_j(\mathbf{e}_k)\mathbf{e}_i, \mathbf{e}_l\rangle + \langle \mathbf{e}_j, \hat{\omega}^i_l(\mathbf{e}_k)\mathbf{e}_i\rangle &(3.2.5)\\ &= \hat{\omega}^l_j(\mathbf{e}_k) + \hat{\omega}^j_l(\mathbf{e}_k), &(3.2.6)\end{aligned}$$

Hence $\hat{\omega}^l_j = -\hat{\omega}^j_l$.

The differential of θ^i can be computed in terms of the Levi-Civita connection and we have that

$$\begin{aligned} d\theta^i(\mathbf{e}_j, \mathbf{e}_k) &= \mathbf{e}_j\theta^i(\mathbf{e}_k) - \mathbf{e}_k\theta^i(\mathbf{e}_j) - \theta^i([\mathbf{e}_j, \mathbf{e}_k])\\ &= -\theta^i([\mathbf{e}_j, \mathbf{e}_k])\\ &= -\theta^i(\nabla_{\mathbf{e}_j}\mathbf{e}_k) + \theta^i(\nabla_{\mathbf{e}_k}\mathbf{e}_j)\\ &= -\theta^i(\hat{\omega}^l_k(\mathbf{e}_j)\mathbf{e}_l) + \theta^i(\hat{\omega}^l_j(\mathbf{e}_k)\mathbf{e}_l)\\ &= \hat{\omega}^i_j(\mathbf{e}_k) - \hat{\omega}^i_k(\mathbf{e}_j)\end{aligned}$$

where the first equality follows from the fact that $\theta^i(\mathbf{e}_k)'s$ are constant and second equality from the fact that ∇ is torsion free connection. But we see that

$$\begin{aligned} (\theta\wedge\hat{\omega})^i(\mathbf{e}_j, \mathbf{e}_k) &= \theta^l\wedge\hat{\omega}^i_l(\mathbf{e}_j, \mathbf{e}_k)\\ &= \theta^l(\mathbf{e}_j)\hat{\omega}^i_l(\mathbf{e}_k) - \theta^l(\mathbf{e}_k)\hat{\omega}^i_l(\mathbf{e}_j)\\ &= \hat{\omega}^i_j(\mathbf{e}_k) - \hat{\omega}^i_k(\mathbf{e}_j).\end{aligned}$$

So

$$d\theta = \theta\wedge\hat{\omega}.$$

Thus the $\hat{\omega}'$s satisfy both condition (a) and (b) of Cartan lemma, so that, by uniqueness, we must have $\hat{\omega}^i_j = \omega^i_j$. ∎

In other word, the lemma shows that $\mathfrak{o}(n)-$valued $1-$form $\omega = (\omega^i_j)$ is the 1 form associated to moving frame X via Levi-Civita connection ∇. In particular, as we show in the following lemma, the 2-forms

$$\Omega = d\omega - \omega\wedge\omega, \quad \Theta = d\theta - \omega\wedge\theta, \tag{3.2.7}$$

which are called *Cartan curvature form* and *torsion form*, respectively are the Riemannian curvature and torsion tensor on the Riemannian manifold. Together the equations (3.2.7) form the *Cartan structure equation.*

Lemma 3.2.5. 1. $\Omega^i_k(\mathbf{e}_l, \mathbf{e}_j) = \langle R(\mathbf{e}_l, \mathbf{e}_j)\mathbf{e}_k, \mathbf{e}_i\rangle$.

2. $\Theta^l(\mathbf{e}_j, \mathbf{e}_k) = \langle T(\mathbf{e}_j, \mathbf{e}_k), \mathbf{e}_l \rangle$.
where R and T are curvature and torsion tensors defined in (3.1.1) and (3.1.6).

Proof. By definition we have

$$\begin{aligned}
R(\mathbf{e}_l, \mathbf{e}_j)\mathbf{e}_k &= \nabla_{\mathbf{e}_l}\nabla_{\mathbf{e}_j}\mathbf{e}_k - \nabla_{\mathbf{e}_j}\nabla_{\mathbf{e}_l}\mathbf{e}_k - \nabla_{[\mathbf{e}_l,\mathbf{e}_j]}\mathbf{e}_k \\
&= \nabla_{\mathbf{e}_l}\omega_k^i(\mathbf{e}_j)\mathbf{e}_i - \nabla_{\mathbf{e}_j}\omega_k^i(\mathbf{e}_l)\mathbf{e}_i - \nabla_{\nabla_{\mathbf{e}_l}\mathbf{e}_j - \nabla_{\mathbf{e}_j}\mathbf{e}_l}\mathbf{e}_k \\
&= +\mathbf{e}_l(\omega_k^i(\mathbf{e}_j))\mathbf{e}_i + \omega_k^i(\mathbf{e}_j)\omega_i^m(\mathbf{e}_l)\mathbf{e}_m \\
&\quad -\mathbf{e}_j(\omega_k^i(\mathbf{e}_l))\mathbf{e}_i - \omega_k^i(\mathbf{e}_l)\omega_i^m(\mathbf{e}_j)\mathbf{e}_m \\
&\quad -(\omega_j^i(\mathbf{e}_l)\omega_k^m(\mathbf{e}_i)\mathbf{e}_m - \omega_l^i(\mathbf{e}_j)\omega_k^m(\mathbf{e}_i)\mathbf{e}_m)
\end{aligned}$$

Hence

$$\begin{aligned}
&\langle R(\mathbf{e}_l, \mathbf{e}_j)\mathbf{e}_k, \mathbf{e}_i \rangle \\
&\qquad = +\mathbf{e}_l(\omega_k^i(\mathbf{e}_j)) + \omega_k^m(\mathbf{e}_j)\omega_m^i(\mathbf{e}_l) \\
&\qquad\quad -\mathbf{e}_j(\omega_k^i(\mathbf{e}_l)) - \omega_k^m(\mathbf{e}_l)\omega_m^i(\mathbf{e}_j) \\
&\qquad\quad -(\omega_j^m(\mathbf{e}_l)\omega_k^i(\mathbf{e}_m) - \omega_l^m(\mathbf{e}_j)\omega_k^i(\mathbf{e}_m))
\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}
\Omega_k^i(\mathbf{e}_l, \mathbf{e}_j) &= (d\omega - \omega \wedge \omega)_k^i(\mathbf{e}_l, \mathbf{e}_j) \\
&= \mathbf{e}_l(\omega_k^i(\mathbf{e}_j)) - \mathbf{e}_j(\omega_k^i(\mathbf{e}_l)) - \omega_k^i([\mathbf{e}_l, \mathbf{e}_j]) \\
&\quad -\omega_k^m \wedge \omega_m^i(\mathbf{e}_l, \mathbf{e}_j) \\
&= \mathbf{e}_l(\omega_k^i(\mathbf{e}_j)) - \mathbf{e}_j(\omega_k^i(\mathbf{e}_l)) - \omega_k^i(\omega_j^m(\mathbf{e}_l)\mathbf{e}_m - \omega_l^m(\mathbf{e}_j)\mathbf{e}_m) \\
&\quad -(\omega_k^m(\mathbf{e}_l)\omega_m^i(\mathbf{e}_j) - \omega_k^m(\mathbf{e}_j)\omega_m^i(\mathbf{e}_l)) \\
&= \mathbf{e}_l(\omega_k^i(\mathbf{e}_j)) - \mathbf{e}_j(\omega_k^i(\mathbf{e}_l)) - (\omega_j^m(\mathbf{e}_l)\omega_k^i(\mathbf{e}_m) - \omega_l^m(\mathbf{e}_j)\omega_k^i(\mathbf{e}_m)) \\
&\quad +\omega_k^m(\mathbf{e}_j)\omega_m^i(\mathbf{e}_l) - \omega_k^m(\mathbf{e}_l)\omega_m^i(\mathbf{e}_j) \\
&= \langle R(\mathbf{e}_l, \mathbf{e}_j)\mathbf{e}_k, \mathbf{e}_i \rangle
\end{aligned}$$

Thus we obtain that

$$\langle R(\mathbf{e}_l, \mathbf{e}_j)\mathbf{e}_k, \mathbf{e}_i \rangle = (d\omega - \omega \wedge \omega)_k^i(\mathbf{e}_l, \mathbf{e}_j).$$

For the proof of the second part, see the comprehensive book written by Michael Spivak [64] volume II. ■

3.3 Ehresmann connection and Cartan geometry

We generalize the idea of classical Cartan connection. The content of this section can be found in various references, Spivak's [64] and Kobayashi's [36] are comprehensive books, Sharpe [63] describes the Cartan generalization of Klein's Erlangen program.

As in Euclidean space there is a natural way to parallel-translate and compare vectors at different points, likewise in a general manifold a choice of a connection prescribes a way of translating tangent vectors "parallel to themselves" and to intrinsically define a directional derivative.

In the case of a principal bundle P with structure group G over a manifold M :

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \downarrow \pi \\ & & M \end{array}$$

We explain the rule of a connection when thinking of lifting a vector field $v \in TM$ to a vector field $\tilde{v} \in TP$ in a unique way. For each $p \in P$, let G_p be the vector subspace of T_pP consisting of all the vectors tangent to the vertical fiber. That is $G_p = ker(d\pi(p)) \subset T_pP$ in which $d\pi(p) : T_pP \to T_{\pi p}M$.

The lifting of v will be unique if we require $\tilde{v}(p)$ to lie in a subspace of T_pP complementary to G_p. A smooth and $G-$invariant choice of such a complementary subspace is called a *Ehresmann connection* (Cf. [17]) on P. This leads to the following definition.

Definition 3.3.1. A connection on a principal bundle P is a smooth assignment of a subspace $H_p \subset T_pP$, for each $p \in P$ such that:

1. $T_pP = G_p \oplus H_p$,
2. $H_{gp} = T_p(L_g)H_p$ for each $g \in G$, where L_g is the left-translation in G and consequently $T_p(L_g) : T_pP \to T_{gp}P$.

Given a connection, the horizontal subspace H_p is mapped isomorphically by $d\pi$ onto $T_{\pi p}M$. Therefore the lifting of v is the unique horizontal $\tilde{v}$ which projects onto v. An equivalent way of assigning a connection is by means of a Lie algebra valued $1-$form ω (Cartan connection). If $X \in \mathfrak{g}$, let $X^\dagger$ be the vector field on P induced by the action of the $1-$parameter subgroup e^{tX}. Since the action of G maps each fiber into itself, then $X^\dagger$ is tangent to the vertical fiber at each point, i.e., $X \in G_p$. For each $v \in T_pP$, we define $\omega(v)$ as the unique $X \in \mathfrak{g}$ such that $X^\dagger$ is equal to the vertical component of v. It follows that $\omega(v) = 0$ if and only if v is horizontal.

Proposition 3.3.1. A Cartan connection $1-$form ω has the following properties:

1. $\omega(X^\dagger) = X$,
2. $L_g^*\omega = \mathrm{Ad}_g\omega$ for each $g \in G$, in which Ad is adjoint representation of G.

The proof can be found in [36] and appendix A of [63].

Now we define the *Cartan geometry* based on the Ehresmann connection. Assume here that H is a group with the Lie algebra $\mathfrak{h}$ as subalgebra of $\mathfrak{g}$.

Definition 3.3.2. A *Cartan geometry* $\xi = (P, \omega)$ on M modeled on $(\mathfrak{g}, \mathfrak{h})$ with group H consist of the following data:

1. a smooth manifold M;
2. a principal left H bundle P over M;
3. a $\mathfrak{g}-$valued $1-$form ω on P satisfies the following conditions:
   (a) for each point $p \in P$, the linear map $\omega_p : T_pP \to \mathfrak{g}$ is an isomorphism;
   (b) $(L_h)^*\omega = \mathrm{Ad}(h)\omega$ for all $h \in H$;
   (c) $\omega(X^\dagger) = X$ for all $X \in \mathfrak{h}$.

The $\mathfrak{g}-$valued form on P given by

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]$$

is called the **curvature**. If $\rho : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ is the canonical projection, then $\rho(\Omega)$ is called the **torsion**. If Ω takes values in the subalgebra $\mathfrak{h}$, we say that the geometry is *torsion free*.

Definition 3.3.3. Let M is a connected manifold. Then Cartan geometry $\xi = (P, \xi)$ has *constant curvature* if $\Omega_p(X_p, Y_p)$ is independent of $p \in P$ whenever the vector fields X and Y are $\omega-$constant vector fields.

That may also be expressed by saying that the curvature function

$$K : P \to \mathrm{Hom}(C^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{h}), \quad K(p) = \Omega_p(\omega_p^{-1}(u), \omega_p^{-1}(v))$$

is constant.

Definition 3.3.4. A Cartan geometry whose curvature vanishes at every point is called *flat*.

Notice that while structure equation always holds for a Lie group, meaning that the curvature of Maurer-Cartan form vanishes, not all Cartan geometry are flat.

3.4 Homogeneous space, symmetric space

The material of this section is taken from [4] and [33]. A homogeneous space of a Lie group G is any differentiable manifold P on which G acts transitively, that is, for $p_1, p_2 \in M$, there is $g \in G$ so that $g.P_1 = p_2$. The subgroup

$$H = H_{p_0} = \{g \in G : g.p_0 = p_0\}$$

is called the isotropy group at p_0. It is a theorem that each such P can be identified with a coset space G/H for some subgroup H and that this H plays the rule of isotropy group of some point.

Let $\mathfrak{g}, \mathfrak{h}$ be the Lie algebras of G and H respectively, and let $\mathfrak{m}$ be the vector space complement of $\mathfrak{h}$ in $\mathfrak{g}$. Then

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \quad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h},$$

and $\mathfrak{m}$ is identified with the tangent space $T_{p_0}M$ of $M = G/H$ at point p_0. At the moment we know nothing of $[\mathfrak{h}, \mathfrak{m}]$ and $[\mathfrak{m}, \mathfrak{m}]$.

Definition 3.4.1. When $\mathfrak{g}$ satisfies the more stringent conditions:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \quad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m},$$

then $M = G/H$ is called a *reductive homogeneous space.*

These spaces possess canonically defined connection with curvature and torsion. Evaluated at fixed point p_0, the curvature and torsion tensors are given purely in terms of the Lie bracket operation on $\mathfrak{m}$:

$$(R(X,Y)Z)_{p_0} = -[[X,Y]_{\mathfrak{h}}, Z], \quad X, Y, Z \in \mathfrak{m},$$

$$T(X,Y)_{p_0} = -[X,Y]_{\mathfrak{m}}, \quad X, Y \in \mathfrak{m},$$

where subscript $\mathfrak{h}$ and $\mathfrak{m}$ refer to the component of $[X,Y]$ in those vector subspaces.

Definition 3.4.2. When $\mathfrak{g}$ satisfies the conditions:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \quad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m},$$

$$[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h},$$

then $\mathfrak{g}$ is called a symmetric algebra and G/H is a symmetric space.

For these spaces the above mentioned canonical connection is derived from a metric, which is itself given by the restriction of the Killing form to $\mathfrak{m}$. Clearly this connection is torsion free and curvature tensor is given as

$$(R(X,Y)Z)_{p_0} = -[[X,Y], Z], \quad X, Y, Z \in \mathfrak{m}.$$

For the symmetric spaces with constant curvature, we do have that

$$\kappa = \frac{K(R(X,Y)Y,X)}{K(X,X)K(Y,Y) - K(X,Y)^2} = -\frac{K([[X,Y],Y],X)}{K(X,X)K(Y,Y) - K(X,Y)^2},$$

for $X, Y \in \mathfrak{m}$.

Remark 3.4.1. There also can be defined a Levi-Civita connection.

Remark 3.4.2. On p.518 of Helgason's book [33] there is a table of symmetric spaces. Directly beneath this table those spaces which are Hermitian are listed.

Remark 3.4.3. The space $Sp(n+1)/Sp(n) \times Sp(1)$ is homogeneous space. There we have that $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$ in which $\mathfrak{g} = \mathfrak{sp}(n+1)$ and $\mathfrak{h} = \mathfrak{sp}(n) \times \mathfrak{sp}(1)$. Moreover this space is naturally reductive space. For definition see [4].