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Chapter 4

Geometric method of Integrable system

4.1 Integrable system in Riemannian geometry

What follows is the description of a *natural frame* and natural formulae (in comparison with the Frenêt frame and the Frenêt formulae) for any smooth curve on a Riemannian manifold under some nondegeneracy conditions that follow from the construction.

Let $\gamma : U \subset \mathbb{R} \to \mathsf{M}$ be a smooth curve on a Riemannian manifold M with Riemannian connection ∇. From now on we will assume that all vector fields are defined on some common open subset U. Let $V(x)$ be the tangent field at x obtained by differentiation with respect to x (also called the velocity vector). We will naturally say that γ is parametrized by arclength whenever $|V(x)| = 1$ for all x in the domain of γ. Assume that γ is nondegenerate, that is, $V(x) \neq 0$ for all $x \in U$. We can then define the first vector in the natural frame, the unit tangent vector, as

$$\mathbf{e}_1(x) = \frac{V(x)}{|V(x)|}.$$

Now as is well known in differential geometry, we can construct the Frenêt frame. More explicitly, let the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ along the curve be Frenêt frame, that is,

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x = \begin{pmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}.$$

The matrix in this equation is called the *Cartan matrix.*

The construction of natural frame is due to the work of Bishop and Hasimoto. In 1975, Bishop, cf. [7] discovered the same transformation as Hasimoto [32] when he studied the relation between two different frames to frame a curve in 3−dimensional Euclidean space. In order to build up the natural frame, let us introduce the following new basis

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \quad \theta = \int k_2.$$

Its frame equation is

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{pmatrix}_x = \begin{pmatrix} 0 & k_1\cos(\theta) & k_1\sin(\theta) \\ -k_1\cos(\theta) & 0 & 0 \\ -k_1\sin(\theta) & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{pmatrix}.$$

We call the basis $\{\mathbf{e}, \mathbf{f}_1, \mathbf{f}_2\}$ the *natural frame* or *parallel frame*.

As a matter of fact, Hasimoto transformation is the one which converts the Frenêt frame into such a natural or parallel frame. This transformation is induced by a gauge transformation, cf. [12, 73] where the Hasimoto transformation has been generalized to arbitrary $n-$dimensional space. In the following definition we have formulated the parallel frame for a curve embedded in a $n-$dimensional Riemannian manifold.

Definition 4.1.1. The natural orthonormal frame $\{\mathbf{e}_i, i = 1, \dots, n\}$ is the frame satisfying the following equation which are called *natural formulae*.

$$\begin{cases} \nabla_{\mathbf{e}_1}\mathbf{e}_1 = -\sum_{i=2}^{n} k_i\mathbf{e}_i \\ \\ \nabla_{\mathbf{e}_1}\mathbf{e}_i = k_i\mathbf{e}_1, \quad i \geq 2, \end{cases} \tag{4.1.1}$$

In the matrix form we have

$$\begin{pmatrix} \nabla_{\mathbf{e}_1}\mathbf{e}_1 \\ \nabla_{\mathbf{e}_1}\mathbf{e}_2 \\ \vdots \\ \nabla_{\mathbf{e}_1}\mathbf{e}_n \end{pmatrix} = \begin{pmatrix} 0 & -k_2 & \dots & -k_n \\ k_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ k_n & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}.$$

Definition 4.1.2. Suppose $\gamma(s,t)$ is any one parameter family of arclength parametrized curves embedded in a Riemannian manifold, and let X be the vector filed $X = \gamma_t$. Then X is locally arclength preserving if $\nabla_{\gamma_s}X$ has no tangential component, i.e.,

$$\langle \nabla_{\gamma_s}X, \gamma_s \rangle = 0.$$

Remark 4.1.1. The vector field X in the above definition is called Sym-Pohlmeyer field in the literature, See for instance [44], [66] and [54].

In the following theorem, using the natural moving frame for the curves, we analyze the correspondence between curve evolution and natural curvature evolution by means of a geometric recursion operator. See [43, lemma in page 81] and [25, theorem 2] in the case of Frenêt moving frame.

Theorem 4.1.2. Let M be $n-$dimensional Riemannian manifold with constant curvature $\varkappa$, and let $\gamma(x,t)$ be family of curves on M satisfying a geometric evolution system of equations of the form

$$\gamma_t = h_1\mathbf{e}_1 + \dots + h_n\mathbf{e}_n, \tag{4.1.2}$$

where $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is the natural frame of γ, and $h_1, \dots, h_n$ are arbitrary smooth functions of the curvatures $k_2, \dots k_n$ and their derivatives with respect to x. Assume that x is the arclength parameter and that evolution (4.1.2) is arclength preserving, that is, $\langle \mathbf{e}_1, \mathbf{e}_1 \rangle = 1$. Then the curvatures $\mathbf{k} = (k_2, \dots, k_n)$ satisfy the evolution

$$\mathbf{k}_t = \mathfrak{H}\mathfrak{I}\mathbf{h} - \varkappa\mathbf{h},$$

where, if we denote by D_x the total differentiation operator with respect to x,

$$\mathfrak{I} = -D_x - D_x^{-1}(< \mathbf{k}, . >)\mathbf{k}, \quad \mathfrak{H} = D_x + \mathfrak{H}_1$$

in which

$$\mathfrak{H}_1\mathbf{g} = D_x^{-1}(\mathbf{k}\mathbf{g}^t - \mathbf{g}\mathbf{k}^t)\mathbf{k}, \quad \text{for} \quad \mathbf{g} = (g_2, \ldots, g_n)^t.$$

Proof. A short comment on the calculations to follow: Let us denote $\mathbf{e}_1 = \gamma_x$, assuming x to be arclength. These vectors are defined as the push-forward vectors of the vectors $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}$, tangent to the domain of γ, through γ. That is, if $\gamma : U \subset \mathbb{R}^2 \to \mathsf{M}$, then $\gamma_t = \gamma_* \frac{\partial}{\partial t}$ acting on functions as $\gamma_t(f) = \frac{\partial}{\partial t} f(\gamma(t,x))$; likewise for x : meaning that $\mathbf{e}_1(f) = \gamma_x(f) = \frac{\partial}{\partial x} f(\gamma(t,x))$ in which f is a function $f : \mathsf{M} \to \mathbb{R}$. Thus, by applying γ_t or γ_x to functions defined along γ, we are indeed taking their derivatives with respect to t or x, respectively. If (4.1.2) is arclength preserving, these two vectors will commute:

$$[\gamma_x, \gamma_t] = 0,$$

since $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial t}$ commute and γ_* preserves Lie bracket.

Now it follows from Definition 3.1.4 of the Riemannian connection and the natural formulae, that

$$\begin{aligned} \gamma_t\langle \nabla_{\mathbf{e}_1}\mathbf{e}_i, \nabla_{\mathbf{e}_1}\mathbf{e}_i\rangle &= 2\langle \nabla_{\gamma_t}\nabla_{\mathbf{e}_1}\mathbf{e}_i, \nabla_{\mathbf{e}_1}\mathbf{e}_i\rangle \\ &= 2\langle \nabla_{\gamma_t}\nabla_{\mathbf{e}_1}\mathbf{e}_i, k_i\mathbf{e}_1\rangle \\ &= 2k_i\langle \nabla_{\gamma_t}\nabla_{\mathbf{e}_1}\mathbf{e}_i, \mathbf{e}_1\rangle, \end{aligned} \tag{4.1.3}$$

for $i = 2, \ldots, n$. On the other hand we can write:

$$\begin{aligned} \gamma_t\langle \nabla_{\mathbf{e}_1}\mathbf{e}_i, \nabla_{\mathbf{e}_1}\mathbf{e}_i\rangle &= \gamma_t\langle k_i\mathbf{e}_1, k_i\mathbf{e}_1\rangle \\ &= \gamma_t(k_i^2) \\ &= 2k_ik_{i,t}, \end{aligned} \tag{4.1.4}$$

where we have denoted $k_{i,t}$ to $\gamma_t(k_i)$. Observe that here is the first place we have used the fact that curve is arclength-preserving, i.e., $\langle \mathbf{e}_1, \mathbf{e}_1\rangle = 1$.

Hence, using the definition of Riemannian curvature tensor, the Riemann-Christoffel curvature tensor and the fact that $\mathbf{e}_1 = \gamma_x$ and γ_t commute, the evolution of k_i can be derived from (4.1.3) and (4.1.4) as follows.

$$
\begin{aligned}
k_{i,t} &= \langle \nabla_{\gamma_t} \nabla_{\mathbf{e}_1} \mathbf{e}_i, \mathbf{e}_1 \rangle \\
&= \langle \nabla_{\mathbf{e}_1} \nabla_{\gamma_t} \mathbf{e}_i + \nabla_{[\mathbf{e}_1, \gamma_t]} \mathbf{e}_i + R(\gamma_t, \mathbf{e}_1)\mathbf{e}_i, \mathbf{e}_1 \rangle \\
&= \langle \nabla_{\mathbf{e}_1} \nabla_{\gamma_t} \mathbf{e}_i, \mathbf{e}_1 \rangle + K(\mathbf{e}_1, \mathbf{e}_i, \gamma_t, \mathbf{e}_1) \\
&= \mathbf{e}_1 \langle \nabla_{\gamma_t} \mathbf{e}_i, \mathbf{e}_1 \rangle - \langle \nabla_{\gamma_t} \mathbf{e}_i, \nabla_{\mathbf{e}_1} \mathbf{e}_1 \rangle + K(\mathbf{e}_1, \mathbf{e}_i, \gamma_t, \mathbf{e}_1) \\
&= \mathbf{e}_1 \langle \nabla_{\gamma_t} \mathbf{e}_i, \mathbf{e}_1 \rangle - \langle \nabla_{\gamma_t} \mathbf{e}_i, -\sum_{l=2}^{n} k_l \mathbf{e}_l \rangle + K(\mathbf{e}_1, \mathbf{e}_i, \gamma_t, \mathbf{e}_1) \\
&= \mathbf{e}_1 \langle \nabla_{\gamma_t} \mathbf{e}_i, \mathbf{e}_1 \rangle + \sum_{l=2}^{n} k_l \langle \nabla_{\gamma_t} \mathbf{e}_i, \mathbf{e}_l \rangle + K(\mathbf{e}_1, \mathbf{e}_i, \gamma_t, \mathbf{e}_1).
\end{aligned}
$$

Let us denote $\langle \nabla_{\gamma_t} \mathbf{e}_i, \mathbf{e}_j \rangle$ by $\omega_i^j(\gamma_t)$ for $i, j = 1, \ldots, n$ according to the Lemma 3.2.4 and Cartan's Lemma 3.2.2. Also let us denote by D_x the total differentiation with respect to x, that is, $D_x = \mathbf{e}_1$. Then the evolution for k_i takes the form:

$$
k_{i,t} = D_x \omega_i^1(\gamma_t) + \sum_{l=2}^{n} k_l \omega_i^l(\gamma_t) + K(\mathbf{e}_1, \mathbf{e}_i, \gamma_t, \mathbf{e}_1) \tag{4.1.5}
$$

In order to compute $\omega_i^l(\gamma_t)$, we use first the fact that the connection is torsion free:

$$
\begin{aligned}
\nabla_{\gamma_t} \mathbf{e}_1 &= [\gamma_t, \mathbf{e}_1] + \nabla_{\mathbf{e}_1} \gamma_t \\
&= \nabla_{\mathbf{e}_1} \gamma_t \\
&= \nabla_{\mathbf{e}_1} (\sum_{i=1}^{n} h_i \mathbf{e}_i) \\
&= \sum_{i=1}^{n} \mathbf{e}_1(h_i) \mathbf{e}_i + \sum_{i=1}^{n} h_i \nabla_{\mathbf{e}_1} \mathbf{e}_i \\
&= (D_x h_1 + \sum_{i=2}^{n} h_i k_i) \mathbf{e}_1 + \sum_{i=2}^{n} (D_x h_i - h_1 k_i) \mathbf{e}_i
\end{aligned} \tag{4.1.6}
$$

Hence since $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}$, so

$$
0 = \omega_1^1(\gamma_t) = \langle \nabla_{\gamma_t} \mathbf{e}_1, \mathbf{e}_1 \rangle = D_x h_1 + \sum_{i=2}^{n} h_i k_i,
$$

and

$$
\langle \nabla_{\gamma_t} \mathbf{e}_1, \mathbf{e}_i \rangle = \omega_1^i(\gamma_t) = D_x h_i - h_1 k_i \quad \text{for} \quad i = 2, \ldots, n.
$$

Hence we can solve the first equation and find h_1 in term of $h_2, \ldots, h_n$, as follows:

$$
h_1 = -D_x^{-1}(< \mathbf{k}, \mathbf{h} >), \quad \text{where} \quad \mathbf{h} = (h_2, \ldots, h_n)^t. \tag{4.1.7}
$$

Notice that the inner product here is just the natural inner product in $\mathbb{R}^{n-1}$.

Also we obtain that

$$\langle \nabla_{\gamma_t}\mathbf{e}_1, \mathbf{e}_i\rangle = \omega_1^i(\gamma_t) = D_x h_i - h_1 k_i \quad \text{for} \quad i = 2, \ldots, n.$$

Hence $\omega_i^1(\gamma_t) = -\omega_1^i(\gamma_t) = -D_x h_i - D_x^{-1}(<\mathbf{k}, \mathbf{h}>)k_i$. We denote by $\mathfrak{I}$ the operator

$$\mathfrak{I} := -D_x - D_x^{-1}(<\mathbf{k}, . >)\mathbf{k} \tag{4.1.8}$$

Now using again the properties of Riemannian connection and Riemannian curvature tensor we can compute the following expression for $i, l \geq 2$:

$$\begin{aligned}
D_x\omega_i^i(\gamma_t) &= \mathbf{e}_1\langle \nabla_{\gamma_t}\mathbf{e}_i, \mathbf{e}_l\rangle \\
&= \langle \nabla_{\mathbf{e}_1}\nabla_{\gamma_t}\mathbf{e}_i, \mathbf{e}_l\rangle + \langle \nabla_{\gamma_t}\mathbf{e}_i, \nabla_{\mathbf{e}_1}\mathbf{e}_l\rangle \\
&= \langle \nabla_{\gamma_t}\nabla_{\mathbf{e}_1}\mathbf{e}_i + R(\mathbf{e}_1, \gamma_t)\mathbf{e}_i, \mathbf{e}_l\rangle + \langle \nabla_{\gamma_t}\mathbf{e}_i, k_l\mathbf{e}_1\rangle \\
&= \langle \nabla_{\gamma_t}(k_i\mathbf{e}_1), \mathbf{e}_l\rangle + K(\mathbf{e}_l, \mathbf{e}_i, \mathbf{e}_1, \gamma_t) + k_l\langle \nabla_{\gamma_t}\mathbf{e}_i, \mathbf{e}_1\rangle \\
&= k_i\langle \nabla_{\gamma_t}\mathbf{e}_1, \mathbf{e}_l\rangle + k_l\langle \nabla_{\gamma_t}\mathbf{e}_i, \mathbf{e}_1\rangle + K(\mathbf{e}_l, \mathbf{e}_i, \mathbf{e}_1, \gamma_t) \\
&= k_i\omega_1^l(\gamma_t) - k_l\omega_1^i(\gamma_t) + K(\mathbf{e}_l, \mathbf{e}_i, \mathbf{e}_1, \gamma_t).
\end{aligned}$$

Hence we simply have that

$$\omega_i^1(\gamma_t) = D_x^{-1}(k_i\omega_1^l(\gamma_t) - k_l\omega_1^i(\gamma_t)) + D_x^{-1}(K(\mathbf{e}_l, \mathbf{e}_i, \mathbf{e}_1, \gamma_t)).$$

Thus, finally, (4.1.5) leads to

$$k_{i,t} = D_x\omega_i^1(\gamma_t) + \sum_{l=2}^{n} D_x^{-1}(k_i\omega_1^l(\gamma_t) - k_l\omega_1^i(\gamma_t))k_l \tag{4.1.9}$$

$$+\sum_{l=2}^{n} D_x^{-1}(K(\mathbf{e}_l, \mathbf{e}_i, \mathbf{e}_1, \gamma_t))k_l + K(\mathbf{e}_1, \mathbf{e}_i, D_t, \mathbf{e}_1). \tag{4.1.10}$$

We use the symbol $\mathfrak{H}$ for the operator

$$\mathfrak{H} := D_x + \mathfrak{H}_1 \tag{4.1.11}$$

where

$$\mathfrak{H}_1\mathbf{g} = D_x^{-1}(\mathbf{k}\mathbf{g}^t - \mathbf{g}\mathbf{k}^t)\mathbf{k}, \quad \text{for} \quad \mathbf{g} = (g_2, \ldots, g_n)^t.$$

Now we simply write the equation for $\mathbf{k}_t$ as follows.

$$\mathbf{k}_t = \mathfrak{H}\mathfrak{I}\mathbf{h} + \sum_{l=2}^{n} k_l D_x^{-1}(K(\mathbf{e}_l, \mathbf{e}_., \mathbf{e}_1, \gamma_t)) + K(\mathbf{e}_1, \mathbf{e}_., \gamma_t, \mathbf{e}_1). \tag{4.1.12}$$

Since the Riemann-Christoffel curvature tensor K in (4.1.12) is multilinear with respect to $C^\infty(\mathsf{M})$ in all its components), it is enough to compute $K(\mathbf{e}_1, \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_1)$ and $K(\mathbf{e}_l, \mathbf{e}_i, \mathbf{e}_1, \mathbf{e}_j)$ for all $i, l = 2, \ldots, n$ and $j = 1, \ldots, n$. We see that

$$K(\mathbf{e}_1, \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_1) = K(\mathbf{e}_j, \mathbf{e}_1, \mathbf{e}_1, \mathbf{e}_i) = 0, \quad \text{for} \quad j \neq 1, i,$$

since $\mathbf{e}_j$ then is perpendicular to $\mathbf{e}_1$ and $\mathbf{e}_i$. On the other hand

$$K(\mathbf{e}_1,\mathbf{e}_i,\mathbf{e}_i,\mathbf{e}_1) = -K(\mathbf{e}_i,\mathbf{e}_1,\mathbf{e}_i,\mathbf{e}_1) = -\sec(\mathbf{e}_i,\mathbf{e}_1) = -\varkappa$$

and

$$K(\mathbf{e}_1,\mathbf{e}_i,\mathbf{e}_1,\mathbf{e}_1) = -K(\mathbf{e}_1,\mathbf{e}_i,\mathbf{e}_1,\mathbf{e}_1) = 0.$$

Thus we have that

$$K(\mathbf{e}_1,\mathbf{e}_i,\mathbf{e}_j,\mathbf{e}_1) = -\varkappa\delta_{ij}, \quad \text{for} \quad i \geq 2, \quad \text{and} \quad j \geq 1.$$

Similarly for all $i, l \neq 1$ and $j \geq 1$, we have

$$K(\mathbf{e}_l,\mathbf{e}_i,\mathbf{e}_1,\mathbf{e}_j) = K(\mathbf{e}_1,\mathbf{e}_j,\mathbf{e}_l,\mathbf{e}_i) = 0.$$

Therefore the equation (4.1.12) becomes as follows:

$$\mathbf{k}_t \quad = \quad \mathfrak{H}\mathfrak{I}\mathbf{h} - \varkappa\mathbf{h} = \mathfrak{R}\mathbf{h} - \varkappa\mathbf{h}. \tag{4.1.13}$$

where the operator $\mathfrak{R}$ is $\mathfrak{R} = \mathfrak{H}\mathfrak{I}$, in which the operators $\mathfrak{I}$ and $\mathfrak{H}$ have been defined in (4.1.8) and (4.1.11), respectively. ■

Remark 4.1.3. The operator

$$\mathcal{P}(h_1\mathbf{e}_1 + h_2\mathbf{e}_2 + \ldots + h_3\mathbf{e}_3) = (-D_x^{-1} < \mathbf{k},\mathbf{h} >)\mathbf{e}_1 + h_2\mathbf{e}_2 + \ldots + h_3\mathbf{e}_3$$

is called the renormalization operator in [44].

Remark 4.1.4. For related topics and similar constructions, see [75, 76, 61, 43].

Remark 4.1.5. Let us now take $\mathbf{h} = \mathbf{k}_x$ in Theorem 4.1.2. Then we obtain

$$\mathbf{k}_t = -\mathbf{k}_3 + \frac{3}{2}\langle\mathbf{k},\mathbf{k}\rangle\mathbf{k}_x - \kappa\mathbf{k}.$$

Indeed we have

$$\mathfrak{I}\mathbf{h} = -\mathbf{k}_{2x} + \frac{1}{2}\langle\mathbf{k},\mathbf{k}\rangle\mathbf{k}, \quad \text{and} \quad \mathfrak{H}_1\mathfrak{I}\mathbf{h} = \mathbf{k}_x\langle\mathbf{k},\mathbf{k}\rangle - \mathbf{k}\langle\mathbf{k}_x,\mathbf{k}\rangle.$$

Theorem 4.1.6. The operators $\mathfrak{H}$ is Hamiltonian, that is, the Schouten bracket $[\mathfrak{H},\mathfrak{H}] = 0$ and $\mathfrak{I}$ is a symplectic operator. The operator $\mathfrak{R}$ is thus a hereditary operator.

Proof. See [58]. ■

Remark 4.1.7. For a curve γ embedded in a Riemannian manifold whose tangent space embedded in a Lie algebra it sufficse to be arclength-preserving, that

$$K(\gamma_s,\gamma_s) = 1,$$

in which K is the Killing form of the Lie algebra.

Remark 4.1.8. One could ask what sort of geometric operators and evolution equations will come up if we use the Frenêt frame? The answer can be found in [25] for nonzero curvature which would be a generalization of [43] in the case of zero curvature. Briefly it was found that evolution of κ, τ (taken as k_1, k_2 here) with respect to time, takes the form

$$\begin{pmatrix}\kappa\\ \tau\end{pmatrix}_t = \hat{P}\begin{pmatrix}h_3\\ h_1\end{pmatrix}$$

where

$$\begin{aligned}\hat{P} &= \begin{pmatrix}-\tau D_x - D_x\tau & D_x^2\frac{1}{\kappa}D_x - \frac{\tau^2}{\kappa}D_x + D_x\kappa\\ D_x\frac{1}{\kappa}D_x^2 - D_x\frac{\tau^2}{\kappa} + \kappa D_x & D_x(\frac{\tau}{\kappa^2} + D_x\frac{\tau}{\kappa^2})D_x + \tau D_x + D_x\tau\end{pmatrix}\\ &\quad +\varkappa\begin{pmatrix}0 & \frac{1}{\kappa}D_x\\ D_x\frac{1}{\kappa} & 0\end{pmatrix}.\end{aligned}$$

Notice the difference between the curvature of the curve κ and sectional curvature $\varkappa$. The division in $1/\kappa$ is consequence of having torsion free connection, see (4.1.6), and solving the corresponding equations dividing by κ. Instead we used nonlocal operator D_x^{-1} as in (4.1.7). Then one can split the operator $\hat{P}$ into the anti-symmetric operators as below.

$$\hat{P} = \mathscr{B} + \mathscr{D} + \mathscr{E} + \varkappa\mathscr{C},$$

in which

$$\mathscr{B} = \begin{pmatrix}-\tau D_x - D_x\tau & -\frac{\tau^2}{\kappa}D_x\\ -D_x\frac{\tau^2}{\kappa} & 0\end{pmatrix}, \quad \mathscr{D} = \begin{pmatrix}0 & D_x\kappa\\ \kappa D_x & \tau D_x + D_x\tau\end{pmatrix},$$

and

$$\mathscr{E} = \begin{pmatrix}0 & D_x^2\frac{1}{\kappa}D_x\\ D_x\frac{1}{\kappa}D_x^2 & D_x(\frac{\tau}{\kappa^2} + D_x\frac{\tau}{\kappa^2})D_x\end{pmatrix}, \quad \mathscr{C} = \begin{pmatrix}0 & \frac{1}{\kappa}D_x\\ D_x\frac{1}{\kappa} & 0\end{pmatrix}.$$

These operators form a quadruplet of compatible Hamiltonians and the Hamiltonian pair $\hat{P}$ and $\mathscr{C}$ gives a hereditary operator $\mathfrak{R}_1 = \hat{P}\mathscr{C}^{-1}$.

Now let us back to Theorem 4.1.2 and evolution of the curve γ in Riemannian manifold M. Corresponding to the natural frame $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, let us denote $\mathbb{R}^n$-valued dual 1-form θ to $\theta = (\theta^1, \dots, \theta^n)$. Then we have that

$$\theta(D_x) = \theta(\mathbf{e}_1) = (1, 0, \dots, 0), \quad \theta(\gamma_t) = (h_1, \dots, h_n), \tag{4.1.14}$$

where $\gamma_t = \sum_{i=1}^{n} h_i \mathbf{e}_i$. Let us denote by D_t the total differentiation with respect to t, that is, $D_t = \gamma_t$. According to the Cartan Lemma 3.2.2, we do have that

$$\nabla_{\mathbf{e}_1}\mathbf{e}_j = \omega_j^k(\mathbf{e}_1)\mathbf{e}_k.$$

This implies that if we let $\omega = (\omega_i^j)_{i,j}$, then in the natural frame (4.1.1), we have that

$$\omega(\mathbf{e}_1) = \begin{pmatrix} 0 & -k_2 & \dots & -k_n \\ k_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ k_n & 0 & \dots & 0 \end{pmatrix}. \tag{4.1.15}$$

As in the proof of Theorem 4.1.2, we could find $\omega(D_t) = \omega(\gamma_t)$ from the the curvature of the Levi-Civita connection ∇ and the fact that the connection is torsion free. Thus we expect to obtain the same equations and geometric operator if we use Cartan structure equations (3.2.7) as we discussed above on the Riemannian manifold modeled on Euclidean geometry. The Cartan structure equation applied to $(\mathbf{e}_1, \gamma_t) = (D_x, D_t)$ gives the following equations:

$$\begin{aligned} \Omega(D_x, D_t) &= (d\omega - \omega \wedge \omega)(D_x, D_t) \\ &= D_x\omega(D_t) - D_t\omega(D_x) - \omega([D_x, D_t]) \\ &\quad -\omega(D_x)\omega(D_t) + \omega(D_t)\omega(D_x) \\ &= D_x\omega(D_t) - D_t\omega(D_x) - \omega(D_x)\omega(D_t) + \omega(D_t)\omega(D_x), \end{aligned}$$

since $[D_x, D_t] = 0$. The torsion form Θ is as follows:

$$\begin{aligned} \Theta(D_x, D_t) &= (d\theta - \omega \wedge \theta)(D_x, D_t) \\ &= D_x\theta(D_t) - D_t\theta(D_x) - T([D_x, D_t]) \\ &\quad -\omega(D_x)\theta(D_t) + \omega(D_t)\theta(D_x) \\ &= D_x\theta(D_t) - D_t\theta(D_x) - \omega(D_x)\theta(D_t) + \omega(D_t)\theta(D_x). \end{aligned}$$

We use the Cartan structure equations:

$$\begin{cases} \Omega(D_x, D_t) = D_x\omega(D_t) - D_t\omega(D_x) - \omega(D_x)\omega(D_t) + \omega(D_t)\omega(D_x) \\ \\ \Theta(D_x, D_t) = D_x\theta(D_t) - D_t\theta(D_x) - \omega(D_x)\theta(D_t) + \omega(D_t)\theta(D_x). \end{cases} \tag{4.1.16}$$

In the case of Riemannian manifold, we have $\Theta(D_x, D_t) = 0$.

Now we replace the instance of $\theta(D_x), \theta(D_t) \in \mathbb{R}^n$ and $\omega(D_x) \in \mathfrak{o}_n$ as in (4.1.14) and (4.1.15) into the Cartan structure (4.1.16) and keep the matrix $\omega(D_t)$ as general element of $\mathfrak{o}_n$ which is to be computed. Let us take the matrix $\omega(D_t)$ as a general element of $\mathfrak{o}_n$ as

$$\omega(D_t) = \begin{pmatrix} 0 & -\mathbf{m}^t \\ \mathbf{m} & \mathbf{M} \end{pmatrix}, \quad \mathbf{M} \in \mathfrak{o}_{n-1} \quad \text{and} \quad \mathbf{m}^t = (m_2, \dots, m_n).$$

With respect to this representation, we write $\omega(D_x)$ as

$$\omega(D_x) = \begin{pmatrix} 0 & -\mathbf{k}^t \\ \mathbf{k} & 0 \end{pmatrix}, \quad \mathbf{k}^t = (k_2, \ldots, k_n).$$

Similarly we write $\theta(D_x)$ and $\theta(D_t)$ as

$$\theta(D_x) = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}, \theta(D_t) = \begin{pmatrix} h_1 \\ \mathbf{h} \end{pmatrix} \quad \mathbf{h}^t = (h_2, \ldots, h_n).$$

Also we write

$$\Omega(D_x, D_t) = \begin{pmatrix} 0 & -\mathbf{r}^t \\ \mathbf{r} & \mathbf{R} \end{pmatrix} \quad \mathbf{R} \in \mathfrak{o}_{n-1} \quad \text{and} \quad \mathbf{r}^t = (r_2, \ldots, r_n).$$

Then the Cartan structure equations (4.1.16) in components will be as follows.

$$\text{Curvature part} : \begin{cases} \mathbf{r} = D_x\mathbf{m} - D_t\mathbf{k} + \mathbf{M}\mathbf{k}, \\ \mathbf{R} = D_x\mathbf{M} + \mathbf{k}\mathbf{m}^t - \mathbf{m}\mathbf{k}^t \end{cases} \tag{4.1.17a}$$

$$\text{Torsion part} : \begin{cases} 0 = D_x h_1 + \langle \mathbf{k}, \mathbf{h} \rangle \\ 0 = D_x\mathbf{h} - \mathbf{k}h_1 + \mathbf{m} \end{cases} \tag{4.1.17b}$$

where $\langle , \rangle$ is standard inner product on $\mathbb{R}^{n-1}$.

Now from the torsion part, we can solve the first equation 4.1.17b and find that

$$h_1 = -D^{-1}\langle \mathbf{k}, \mathbf{h} \rangle.$$

Then the second one can be solved for $\mathbf{m}$ as $\mathbf{m} = -D_x\mathbf{h} - \mathbf{k}D^{-1}\langle \mathbf{k}, \mathbf{h} \rangle = \mathfrak{J}\mathbf{h}$, where the operator $\mathfrak{J} = -D_x - \mathbf{k}D^{-1}\langle \mathbf{k}, . \rangle$ is exactly the operator described in (4.1.8) and proved to be symplectic.

From the curvature part, we see that

$$\mathbf{M} = D^{-1}(\mathbf{m}\mathbf{k}^t - \mathbf{k}\mathbf{m}^t) + D^{-1}(\mathbf{R}).$$

This implies that

$$\begin{aligned} D_t\mathbf{k} &= D_x\mathbf{m} + \mathbf{M}\mathbf{k} - \mathbf{r} \\ &= D_x\mathbf{m} + D_x^{-1}(\mathbf{m}\mathbf{k}^t - \mathbf{k}\mathbf{m}^t)\mathbf{k} + D^{-1}(\mathbf{R})\mathbf{k} - \mathbf{r} \\ &= \mathfrak{H}\mathbf{m} + D^{-1}(\mathbf{R})\mathbf{k} - \mathbf{r} \end{aligned}$$

where $\mathfrak{H} = D_x + \mathfrak{H}_1$ is operator (4.1.11) which is proved to be Hamiltonian. Now similar to the discussion at the end of the proof of Theorem 4.1.2, we obtain $\mathbf{R} = 0$ and $\mathbf{r} = \varkappa\mathbf{h}$. Therefore

$$D_t\mathbf{k} = \mathfrak{R}\mathbf{h} - \varkappa\mathbf{h},$$

in which

$$\mathfrak{R} = \mathfrak{H}\mathfrak{J}.$$

Remark 4.1.9. If one compares the computational efforts of setting up a moving Frenêt frame and structure equations (Levi-Civita connection) just using the metric with the Cartan formulation in terms of connection, the difference is striking: not only one can not see what one is doing, but there is no need to see it, since every thing goes right by construction, and the entities one writes down are automatically differential invariants. See [59].

Remark 4.1.10. One can start with the Euclidean Lie algebra $\mathfrak{g} = \mathfrak{euc}_n(\mathbb{R}) = \mathfrak{o}_n(\mathbb{R}) \ltimes \mathbb{R}^n$ and $\mathfrak{h} = \mathfrak{o}_n$ and assume that ω is the Cartan connection in Euclidean geometry as a Cartan geometry. Let us choose $\omega(D_x)$ and $\omega(D_t)$ as follows.

$$\omega(D_x) = \begin{pmatrix} 0 & -\mathbf{k}^t & 1 \\ \mathbf{k} & 0 & 0^t \\ 0 & 0 & 0 \end{pmatrix}, \quad \omega(D_t) == \begin{pmatrix} 0 & -\mathbf{m}^t & h_1 \\ \mathbf{m} & \mathbf{M} & \mathbf{h}^t \\ 0 & 0 & 0 \end{pmatrix}.$$

Now if we write down the Cartan structure equation

$$\Omega(D_x, D_t) = D_x\omega(D_t) - D_t\omega(D_x) + [\omega(D_t), \omega(D_x)],$$

then we will find exactly the same formula for evolution of $\mathbf{k}$.

4.2 Integrable system in (p,q)-Orthogonal geometry

We extend the idea of the last section in finding the interaction between geometry in a sense of Cartan and integrable system. The direct generalization of Euclidean geometry as an example of a Cartan geometry to the Riemannian geometry of signature p, q will be defined as follows.

Definition 4.2.1. Let

$$\Sigma_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -\epsilon I_q \end{pmatrix}.$$

Then define the orthogonal group of signature p, q by

$$O_{p,q}(\mathbb{R}) = \{A \in Gl_{p+q}(\mathbb{R}) | A\Sigma_{p,q}A^t = \Sigma_{p,q}\}.$$

The Lie algebra of this Lie group is

$$\mathfrak{o}_{p,q}(\mathbb{R}) = \{A \in M_{p+q}(\mathbb{R}) | A\Sigma_{p,q} + \Sigma_{p,q}A^t = 0\}.$$

Any elements of $\mathfrak{o}_{p,q}(\mathbb{R})$ has the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A \in M(p,p), B \in M(p,q), C \in M(q,p)$ and $D \in M(q,q)$ and that

$$A + A^t = D + D^t = 0, \quad B^t = \epsilon C.$$

The reason we use ϵ in the definition of this Lie group is that we can keep trace of the Riemannian case and see how is consistent with that of last section by taking $\epsilon = -1$ and of course to standard geometry of orthogonal group of signature p, q by taking $\epsilon = 1$.

Definition 4.2.2. The Riemannian geometry of signature p, q in a sense of Klein geometry, see Sharpe [63] definition 3.2., is described as a pair of Lie groups (G, H) in which

$$G = O_{p,q}(\mathbb{R}) \ltimes \mathbb{R}^{p+q}, \quad H = O_{p,q}(\mathbb{R}).$$

Similar to the Riemannian case, we first choose a moving frame namely parallel frame and write the Cartan structure equation having Cartan connection ω with free torsion. Notice that the Riemannian manifold $\mathsf{M} = G/H$ is assumed to have zero constant curvature. The parallel moving frame is described as

$$\omega(D_x) = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \begin{pmatrix} 0 & u_1 & \dots & u_{p-1} & u_p & \dots & u_{p+q-1} \\ -u_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -u_{p-1} & 0 & \dots & 0 & 0 & \dots & 0 \\ \epsilon u_p & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \epsilon u_{p+q-1} & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix},$$

as an element of $\mathfrak{o}_{p,q}(\mathbb{R})$.

Remark 4.2.1. As in the Riemannian case, we consider x as arclength parameter, so that the number of invariants are $p + q - 1$. In general this number is just $\dim(\mathsf{M}) - 1$. In the sequel, we take this consideration into account.

Now let $\omega(D_t)$ be the matrix

$$\omega(D_t) = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

in which the matrices $A \in M(p,p), B \in M(p,q), C \in M(q,p)$ and $D \in M(q,q)$, possess the following properties:

$$A + A^t = D + D^t = 0, \quad B^t = \epsilon C.$$

As in the Riemannian case, let the torsion 1-form at D_x and D_t be

$$\tau(D_x) = (1, 0, \dots, 0, 0, \dots, 0)^t,$$

$$\tau(D_t) = (h_1, h_2, \dots, h_{p-1}, h_p, \dots, h_{p+q})^t,$$

respectively.

Theorem 4.2.2. With the assumptions above, the Cartan struction equations as in (4.1.16) gives the evolution of $\mathbf{u}^t = (u_1, \dots, u_{p+q-1})$ as below.

$$D_t\mathbf{u} \;=\; \mathfrak{R}(I^{\epsilon}_{p-1,q}\mathbf{h}), \quad \mathfrak{R} = \mathfrak{H}\mathfrak{I}, \tag{4.2.1}$$

in which

$$\mathbf{h}^t = (h_2, \ldots, h_{p+q}), \quad I^{\epsilon}_{p-1,q} = \begin{pmatrix} I_{p-1} & 0 \\ 0 & -\epsilon I_q \end{pmatrix},$$

and

$$\mathfrak{H} = D_x + \mathfrak{H}_1, \quad \mathfrak{I} = -D_x - \mathbf{u} D_x^{-1} < \mathbf{u}, . >_{pq},$$

where

$$\mathfrak{H}_1 \mathbf{w} = (D_x^{-1}(\mathbf{w}(I^{\epsilon}_{p-1,q}\mathbf{u})^t - \mathbf{u}(I^{\epsilon}_{p-1,q}\mathbf{w})^t)\mathbf{u}, \quad \text{for} \quad \mathbf{w}^t = (w_1, \ldots, w_{p+q-1}),$$

and

$$< \mathbf{u}, \mathbf{h} >_{pq} = < \mathbf{u}, I^{\epsilon}_{p-1,q}\mathbf{h} > .$$

The inner product on the right hand side is the normal inner product on $\mathbb{R}^{p+q-1}$.

Proof. Let us first compute the multiplication type terms in the Cartan structure equation (4.1.16) and find that $\omega(D_t)\tau(D_x) = (A_{11}, A_{21}, \ldots, A_{p1}, C_{11}, \ldots, C_{q1})^t$ and

$$\omega(D_x)\tau(D_t) = (\sum_{k=1}^{p+q-1} u_k h_{k+1}, -u_1 h_1, -u_2 h_1, \ldots, -u_{p-1} h_1, \epsilon u_p h_1, \ldots, \epsilon u_{p+q-1} h_1)^t.$$

Now we use the Cartan structure equation applied to (D_x, D_t) as described in (4.1.16) and obtain the following equations of which the first group is concerned with the curvature part, and the second one with the free torsion part.

$$D_x A_{i1} + D_t u_{i-1} - \sum_{k=2}^{p} A_{ik} u_{k-1} + \sum_{k=1}^{q} B_{ik} \epsilon u_{k+p-1} = 0, \tag{4.2.2a}$$

for $\quad 1 < i \le p,$

$$D_x A_{ij} + A_{i1} u_{j-1} - A_{j1} u_{i-1} = 0, \quad \text{for} \quad 1 < i, j \le p \tag{4.2.2b}$$

$$D_x B_{1j} - D_t u_{p+j-1} - \sum_{k=2}^{p} u_{k-1} B_{kj} - \sum_{k=1}^{q} u_{k+p-1} D_{kj} = 0, \tag{4.2.2c}$$

for $\quad 1 \le j \le q,$

$$D_x B_{ij} + A_{i1} u_{p+j-1} + u_{i-1} B_{1j} = 0, \tag{4.2.2d}$$

for $\quad 1 < i \le p, 1 \le j \le q,$

$$D_x D_{ij} + C_{i1} u_{p+j-1} - C_{j1} u_{p+i-1} = 0, \quad \text{for} \quad 1 \le i, j \le q. \tag{4.2.2e}$$

$$D_x h_1 - \sum_{k=1}^{p+q-1} u_k h_{k+1} = 0 \tag{4.2.3a}$$

$$D_x h_i + u_{i-1} h_1 + A_{i1} = 0 \quad \text{for} \quad 1 < i \le p \tag{4.2.3b}$$

$$D_x h_i - \epsilon u_{i-1} h_1 + C_{(i-p)1} = 0 \quad \text{for} \quad p+1 \le i \le q+p \tag{4.2.3c}$$

Now from (4.2.2d) and the fact that $B_{1j} = \epsilon C_{j1}$, we find that

$$B_{ij} = -D_x^{-1}(A_{i1} u_{p+j-1} + \epsilon u_{i-1} C_{j1}).$$

Also from Equations (4.2.2b) and (4.2.2e) we obtain

$$A_{ij} = D_x^{-1}(A_{j1}u_{i-1} - A_{i1}u_{j-1}), \quad D_{ij} = D_x^{-1}(C_{j1}u_{p+i-1} - C_{i1}u_{p+j-1}).$$

Hence plugging A_{ij}, B_{ij} and D_{ij} into (4.2.2a) and (4.2.2c), one can derive the evolution of u_{i-1} for $i = 2, \ldots p$ and u_{p+j-1} for $j = 1, \ldots q$ respectively as below:

$$\begin{aligned} D_t u_{i-1} &= -D_x A_{i1} + \sum_{k=2}^{p} A_{ik} u_{k-1} - \sum_{k=1}^{q} B_{ik} \epsilon u_{k+p-1} \\ &= -D_x A_{i1} + \sum_{k=2}^{p} u_{k-1} D_x^{-1}(A_{k1} u_{i-1} - A_{i1} u_{k-1}) \\ &\quad + \sum_{k=1}^{q} \epsilon u_{k+p-1} D_x^{-1}(A_{i1} u_{p+k-1} + \epsilon u_{i-1} C_{k1}), \end{aligned} \tag{4.2.4}$$

$$\begin{aligned} D_t u_{p+j-1} &= D_x B_{1j} - \sum_{k=2}^{p} u_{k-1} B_{kj} - \sum_{k=1}^{q} u_{k+p-1} D_{kj} \\ &= \epsilon D_x C_{j1} + \sum_{k=2}^{p} u_{k-1} D_x^{-1}(A_{k1} u_{p+j-1} + \epsilon u_{k-1} C_{j1}) \\ &\quad - \sum_{k=1}^{q} u_{k+p-1} D_x^{-1}(C_{j1} u_{p+k-1} - C_{k1} u_{p+j-1}). \end{aligned} \tag{4.2.5}$$

Hence the evolution equation can be rewritten as follows:

$$D_t \mathbf{u} = -(D_x + \mathfrak{H}_1) I^{\epsilon}_{p-1,q} \mathbf{w} = -\mathfrak{H} I^{\epsilon}_{p-1,q} \mathbf{w}, \tag{4.2.6}$$

where $\mathbf{w} = (A_{21}, \ldots, A_{p1}, C_{11}, \ldots, C_{q1})^t$.

Now (4.2.3a) of the torsion part gives us

$$h_1 = D_x^{-1}\left(\sum_{k=1}^{p+q-1} u_k h_{k+1}\right). \tag{4.2.7}$$

Using the inner product defined on $\mathbb{R}^{p+q-1}$ we can write h_1 as $h_1 = D_x^{-1} < \mathbf{u}, \mathbf{h} >$ in which $\mathbf{h} = (h_2, \ldots h_{p+q})^t$. Also we can derive A_{i1} for $1 < i \leq p$ and $C_{(i-p)1}$ for $p+1 \leq i \leq q+p$ from the equations (4.2.3a) and (4.2.3a) of the torsion part as follows:

$$\begin{aligned} A_{i1} &= -D_x h_i - u_{i-1} h_1 \quad \text{for} \quad 1 < i \leq p, \\ C_{(i-p)1} &= \epsilon u_{i-1} h_1 - D_x h_i \quad \text{for} \quad p+1 \leq i \leq q+p. \end{aligned}$$

These equations together with (4.2.7) give us the formulae for the vector $\mathbf{w}$ in terms of $\mathbf{h} = (h_2, \ldots h_{p+q})^t$ as below:

$$\mathbf{w} = -D_x \mathbf{h} - I^{\epsilon}_{p-1,q} \mathbf{u} D_x^{-1} < \mathbf{u}, \mathbf{h} >,$$

from which we simply find that

$$\begin{aligned} I^{\epsilon}_{p-1,q}\mathbf{w} &= -D_x I^{\epsilon}_{p-1,q}\mathbf{h} - \mathbf{u}D_x^{-1} < \mathbf{u},\mathbf{h} > \\ &= (-D_x - \mathbf{u}D_x^{-1} < \mathbf{u}, . >_{pq})(I^{\epsilon}_{p-1,q}\mathbf{h}) \\ &= \mathfrak{I}(I^{\epsilon}_{p-1,q}\mathbf{h}) \end{aligned}$$

Hence we can write the evolution (4.2.6) as follows:

$$\begin{aligned} D_t\mathbf{u} &= -(D_x + \mathfrak{H}_1)I^{\epsilon}_{p-1,q}\mathbf{w} \\ &= -\mathfrak{H}I^{\epsilon}_{p-1,q}\mathbf{w} \\ &= \mathfrak{H}\mathfrak{I}(I^{\epsilon}_{p-1,q}\mathbf{h}). \end{aligned}$$

■

Let us replace $\mathbf{h}$ by the trivial symmetry $\mathbf{u}_x$, and compute the evolution equation (4.2.1). First we see that

$$\mathfrak{I}(I^{\epsilon}_{p-1,q}\mathbf{u}_x) = -I^{\epsilon}_{p-1,q}\mathbf{u}_{2x} - \frac{1}{2}\mathbf{u} < \mathbf{u},\mathbf{u} > .$$

Then we find that

$$\mathfrak{H}_1\mathfrak{I}(I^{\epsilon}_{p-1,q}\mathbf{u}_x) = \mathbf{u} < \mathbf{u}_x,\mathbf{u} > -\mathbf{u}_x < \mathbf{u},\mathbf{u} > .$$

Hence we find the evolution equation:

$$\begin{aligned} D_t\mathbf{u} &= -I^{\epsilon}_{p-1,q}\mathbf{u}_{3x} - \frac{1}{2}\mathbf{u}_x < \mathbf{u},\mathbf{u} > -\mathbf{u} < \mathbf{u}_x,\mathbf{u} > \\ &\quad \mathbf{u} < \mathbf{u}_x,\mathbf{u} > -\mathbf{u}_x < \mathbf{u},\mathbf{u} > \\ &= -I^{\epsilon}_{p-1,q}\mathbf{u}_{3x} - \frac{3}{2}\mathbf{u}_x < \mathbf{u},\mathbf{u} > . \end{aligned}$$

This is well known mKDV equation type.

Theorem 4.2.3. The operators $\mathfrak{H}, \mathfrak{I}$ and $\mathfrak{R}$ are Hamiltonian, symplectic and hereditary operator respectively.

Remark 4.2.4. Notice that the inner product $< \mathbf{u},\mathbf{h} >_{pq}$ is nothing but the Killing form of two specific matrices in the Lie algebra $\mathfrak{o}_{p,q}(\mathbb{R})$, namely

$$\begin{pmatrix} 0 & \hat{\mathbf{u}}^t & \tilde{\mathbf{u}}^t \\ -\mathbf{u} & 0 & 0 \\ \epsilon\mathbf{u} & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \hat{\mathbf{h}}^t & \tilde{\mathbf{h}}^t \\ -\mathbf{h} & 0 & 0 \\ \epsilon\mathbf{h} & 0 & 0 \end{pmatrix},$$

where for instance $\hat{\mathbf{u}}^t = (u_1, \ldots, u_{p-1})$ and $\tilde{\mathbf{u}}^t = (u_p, \ldots, u_{p+q-1})$, likewise for $\hat{\mathbf{h}}^t$ and $\tilde{\mathbf{h}}^t$.

Remark 4.2.5. One can compare the operator $\mathfrak{H}$ presented here and the one derived in [58] in the case of Riemannian geometry. In fact we do have that

$$\mathfrak{H}\mathbf{w} = \sum_{i<j} J'_{ij}\mathbf{u}D^{-1} < J'_{ij}\mathbf{u},\mathbf{w} >_{pq},$$

in which $J'_{ij} = J_{ij}I^{\epsilon}_{p-1,q}$ and $(J_{ij})_{kl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$.

4.3 Integrable system in symplectic geometry

We consider the group $G = Sp(n+2)$ over the division ring of quaternions $\mathbb{H}$ of all invertible matrix in $GL(n+2, \mathbb{H})$ as defined in Chapter 1.

Now let H be the closed subgroup $SP(1) \times SP(n+1)$ of G. In fact $SP(1)$ sits in the first diagonal entry of G and $SP(n+1)$ sits in the diagonal block left. Let us denote the Lie algebra of G by $\mathfrak{g}$ and that of H by $\mathfrak{h}$. As is known in the literature, $\mathsf{M} = G/H$ is then a smooth manifold. In fact M is the quaternionic projective space $\mathbb{HP}^{n-1}$. For a comprehensive reference, see [63].

Similar to the situation in Riemannian manifold [58], given a curve in M, we know its tangent vectors D_x and want to compute all possible D_t. Let ω be Cartan 1−form with its values in the Lie algebra $\mathfrak{g}$. We make a specific choice of $\omega(D_x)$ and leave $\omega(D_t)$ as a general element of $\mathfrak{g}$. We see that the dimension of M is equal to the dimension of $\mathfrak{g}/\mathfrak{h}$ which is easily computed to be $4n$. With x taken to be the arc length parameter, the dimension of the space of differential invariants in $\omega(D_x)$ describing the curve must be one less the dimension of the manifold, that is, $4n-1$, see Remark 4.2.1.

Instead of working on curvature and torsion separately, we can put them in one picture. That means we can increase the dimension by one and put $\tau(D_x)$ in first row of $\omega(D_x)$. Following this schema, we will have $\omega(D_t)$ as general element of $SP(n+2)$.

Now let us choose a Cartan matrix $\omega(D_x)$ similar to that of the parallel coframe in Riemannian geometry with proper dimension counting as follows:

$$\omega(D_x) = \begin{pmatrix} 0 & 1 & \mathbf{0}^t \\ -1 & u & -\overline{\mathbf{u}}^t \\ \mathbf{0} & \mathbf{u} & \mathbf{0} \end{pmatrix},$$

where $\omega(D_x)$ is taken as an element of $\mathfrak{sp}(n+2)$.

Remark 4.3.1. Important notice should be taken into consideration that u is purely imaginary, and $\mathbf{u} \in \mathbb{H}^{n-1}$ following the fact that $\omega(D_x)$ is in $\mathfrak{sp}(n+2)$.

Remark 4.3.2. Other choices of coframe tend to destroy the scalar-vector character of the analysis and complicate matters tremendously, which is one of the main reasons why the n-dimensional analysis using Frenêt frames seems to be out of reach.

We see that this matrix is parametrized by $4n-1$ real parameters. Notice that here we have taken the curvature and torsion part of Cartan form in one picture.

Now $\omega(D_t)$ must be a typical element of $\mathfrak{g}$ which we write as follows:

$$\omega(D_t) = \begin{pmatrix} m_{11} & m_{12} & -\overline{\mathbf{m}}_1^t \\ m_{21} & m_{22} & -\overline{\mathbf{m}}_2^t \\ \mathbf{m}_1 & \mathbf{m}_2 & \mathbf{M} \end{pmatrix}.$$

In the Riemannian case, if we use a parallel frame and assume constant curvature $\varkappa$, this can be taken zero and we still can derive all the geometric quantities. Therefore we have taken the curvature equal to zero. Hence the Cartan structure equation evaluated at the evolutionary vector fields D_x, D_t is as follows:

$$D_x\omega(D_t) - D_t\omega(D_x) + [\omega(D_t), \omega(D_x)] = 0.$$

Before we explore the Cartan structure equation, let us define some notation. Commutators of vectors and scalars are defined by

$$C_{\mathbf{u}}\mathbf{m}_2 := \langle \mathbf{u}, \mathbf{m}_2\rangle - \langle \mathbf{m}_2, \mathbf{u}\rangle, \quad C_u m_{22} := u m_{22} - m_{22} u,$$

where the inner product $\langle \cdot, \cdot \rangle$ is the Hermitian inner product. Right multiplication by scalar u on vector $\mathbf{h}$ and left multiplication by vector $\mathbf{u}$ on scalar h are defined respectively by

$$R_u\mathbf{h} = \mathbf{h}u, \quad L_{\mathbf{u}}h = \mathbf{u}h.$$

On the other hand, the anti-commutators on vector and scalar quantities are defined by

$$A_{\mathbf{u}}\mathbf{h} = \langle \mathbf{u}, \mathbf{h}\rangle + \langle \mathbf{h}, \mathbf{u}\rangle, \quad A_u h = uh + hu.$$

Now we explicitly write the components of the Cartan structure equation. Among these equations, the four first equations are concerned with the curvature and the last three with the torsion. These equations lead to evolution of the scalar invariant u and the vector invariant $\mathbf{u}$ as combination of geometric operators applied on the proper torsion variables of $\omega(D_t)$ according to the proposition below, in which we have defined $\mathfrak{H}_1$ as the operator acting on vectors by

$$\mathfrak{H}_1\mathbf{h} = \left(D_x^{-1}(\mathbf{h}\overline{\mathbf{u}}^{\,t} - \mathbf{u}\overline{\mathbf{h}}^{\,t})\right)\mathbf{u}, \tag{4.3.1}$$

where, for instance, $\mathbf{h}\overline{\mathbf{u}}^{\,t}$ is the outer product of a vector and a covector, that is, a matrix. As we have seen before, this operator appear also in the case of Riemannian geometry of signature p, q, see Theorem 4.2.2. Hence, for instance, we can write $\mathbf{M}\mathbf{u} = \mathfrak{H}_1\mathbf{m}_2$. when we see in (4.3.2c) below.

$$\begin{aligned}
&D_x m_{11} - m_{12} - m_{21} = 0 && \text{(4.3.2a)}\\
&D_x m_{22} - D_t u - C_u m_{22} + C_{\mathbf{u}}\mathbf{m}_2 + m_{12} + m_{21} = 0 && \text{(4.3.2b)}\\
&D_x\mathbf{m}_2 - D_t\mathbf{u} + R_u\mathbf{m}_2 + \mathfrak{H}_1\mathbf{m}_2 - L_{\mathbf{u}}m_{22} + \mathbf{m}_1 = 0 && \text{(4.3.2c)}\\
&D_x\mathbf{M} - \mathbf{m}_2\overline{\mathbf{u}}^{\,t} + \mathbf{u}\overline{\mathbf{m}}_2^{\,t} = 0 && \text{(4.3.2d)}\\
&D_x\mathbf{m}_1 - \mathbf{m}_2 - \mathbf{u}m_{21} = 0 && \text{(4.3.2e)}\\
&D_x m_{12} + m_{11} - m_{22} + m_{12}u - \langle \mathbf{m}_1, \mathbf{u}\rangle = 0 && \text{(4.3.2f)}\\
&D_x m_{21} + m_{11} - m_{22} - u m_{21} + \langle \mathbf{u}, \mathbf{m}_1\rangle = 0 && \text{(4.3.2g)}
\end{aligned}$$

Solving these equations we obtain

Theorem 4.3.3. The evolution of differential invariants can be written in the form

$$\begin{pmatrix} D_t u \\ D_t\mathbf{u} \end{pmatrix} = \mathfrak{H}\mathfrak{I}\begin{pmatrix} m_{12} + m_{21} \\ \mathbf{m}_1 \end{pmatrix} + \mathfrak{A}\begin{pmatrix} m_{12} + m_{21} \\ \mathbf{m}_1 \end{pmatrix}, \tag{4.3.3}$$

where

$$\mathfrak{H} = \begin{pmatrix} D_x - C_u & C_{\mathbf{u}} \\ -L_{\mathbf{u}} & D_x + R_u + \mathfrak{H}_1 \end{pmatrix}, \quad \mathfrak{A} = \begin{pmatrix} (2D_x - C_u)D_x^{-1} & 0 \\ -L_{\mathbf{u}}D_x^{-1} & I \end{pmatrix},$$

and

$$\mathfrak{I} = \begin{pmatrix} \frac{1}{2}D_x - \dfrac{1}{4}C_u - \dfrac{1}{4}A_u D_x^{-1}\frac{1}{2}A_u & \frac{1}{2}C_{\mathbf{u}} + \frac{1}{2}uD_x^{-1}A_{\mathbf{u}} \\ -\frac{1}{2}L_{\mathbf{u}}D_x^{-1}\frac{1}{2}A_u - \frac{1}{2}L_{\mathbf{u}} & D_x + \frac{1}{2}L_{\mathbf{u}}D_x^{-1}A_{\mathbf{u}} \end{pmatrix}.$$

Proof. We start with (4.3.2b) and (4.3.2c) to find the evolution $\begin{pmatrix} D_t u \\ D_t \mathbf{u} \end{pmatrix}$ as presented below.

$$\begin{aligned} \begin{pmatrix} D_t u \\ D_t \mathbf{u} \end{pmatrix} &= \begin{pmatrix} D_x - C_u & C_{\mathbf{u}} \\ -L_{\mathbf{u}} & D_x + \hat{R}_u + \mathfrak{H}_1 \end{pmatrix} \begin{pmatrix} m_{22} \\ \mathbf{m}_2 \end{pmatrix} + \begin{pmatrix} m_{12} + m_{21} \\ \mathbf{m}_1 \end{pmatrix} \\ &= \mathfrak{H} \begin{pmatrix} m_{22} \\ \mathbf{m}_2 \end{pmatrix} + \begin{pmatrix} m_{12} + m_{21} \\ \mathbf{m}_1 \end{pmatrix}. \end{aligned} \quad (4.3.4)$$

Now from (4.3.2f) and (4.3.2g) which are actually the torsion part, we obtain $m_{11} - m_{22}$ in terms of m_{12}, m_{21} and $\mathbf{m}_1$. Hence in the evolution (4.3.4) we subtract and add m_{11} in a proper way.

$$\begin{aligned} \begin{pmatrix} D_t u \\ D_t \mathbf{u} \end{pmatrix} &= \mathfrak{H} \begin{pmatrix} m_{22} - m11 \\ \mathbf{m}_2 \end{pmatrix} + \mathfrak{H} \begin{pmatrix} m_{11} \\ 0 \end{pmatrix} + \begin{pmatrix} m_{12} + m_{21} \\ \mathbf{m}_1 \end{pmatrix} \\ &= \mathfrak{H} \begin{pmatrix} m_{22} - m_{11} \\ \mathbf{m}_2 \end{pmatrix} + \begin{pmatrix} D_x - C_u & 0 \\ -L_{\mathbf{u}} & 0 \end{pmatrix} \begin{pmatrix} m_{11} \\ \mathbf{m}_1 \end{pmatrix} + \begin{pmatrix} m_{12} + m_{21} \\ \mathbf{m}_1 \end{pmatrix} \end{aligned} \quad (4.3.5)$$

If we subtract (4.3.2f) from (4.3.2g), we deduce that

$$D_x(m_{21} - m_{12}) = u m_{21} + m_{12} u - A_{\mathbf{u}} \mathbf{m}_1.$$

Using the fact that $\frac{1}{2}A_u(m_{12} + m_{21}) = m_{12}u + um_{21}$ we obtain that

$$m_{21} - m_{12} = D_x^{-1}(\frac{1}{2}A_u(m_{12} + m_{21}) - A_{\mathbf{u}}\mathbf{m}_1). \quad (4.3.6)$$

Taking the difference of (4.3.2f) and (4.3.2g), we derive an expression for $m_{22} - m_{11}$:

$$m_{22} - m_{11} = \frac{1}{2}D_x(m_{12} + m_{21}) + \frac{1}{2}C_{\mathbf{u}}\mathbf{m}_1 + \frac{1}{2}(m_{12}u - um_{21}).$$

Using the fact that $m_{12}u - um_{21} = -\frac{1}{2}C_u(m_{12} + m_{21}) + u(m_{12} - m_{21})$ and (4.3.6) enable us to convert last equation to the following one.

$$\begin{aligned} m_{22}-m_{11} &= \frac{1}{2}D_x(m_{12}+m_{21})+\frac{1}{2}C_{\mathbf{u}}\mathbf{m}_1 \\ &\quad -\frac{1}{4}C_u(m_{12}+m_{21})-\frac{1}{2}uD_x^{-1}(\frac{1}{2}A_u(m_{12}+m_{21})-A_{\mathbf{u}}\mathbf{m}_1). \end{aligned}$$

On the other hand, we can write m_{21} as below.

$$\begin{aligned} m_{21} &= \frac{m_{21}-m_{12}}{2}+\frac{m_{21}+m_{12}}{2} \\ &= \frac{1}{2}D_x^{-1}(\frac{1}{2}A_u(m_{12}+m_{21})-A_{\mathbf{u}}\mathbf{m}_1)+\frac{m_{21}+m_{12}}{2}. \end{aligned}$$

Hence by using (4.3.2e), we can express $\begin{pmatrix} m_{22}-m_{11} \\ \mathbf{m}_2 \end{pmatrix}$ as follows.

$$\begin{aligned} &\begin{pmatrix} m_{22}-m_{11} \\ \mathbf{m}_2 \end{pmatrix} \\ =& \begin{pmatrix} \frac{1}{2}D_x-\frac{1}{4}C_u-\frac{1}{2}uD^{-1}\frac{1}{2}A_u & \frac{1}{2}C_{\mathbf{u}}+\frac{1}{2}uD_x^{-1}A_{\mathbf{u}} \\ -\frac{1}{2}L_{\mathbf{u}}D_x^{-1}\frac{1}{2}A_u-\frac{1}{2}L_{\mathbf{u}} & D_x+\frac{1}{2}L_{\mathbf{u}}D_x^{-1}A_{\mathbf{u}} \end{pmatrix}\begin{pmatrix} m_{12}+m_{21} \\ \mathbf{m}_1 \end{pmatrix} \\ =& \mathfrak{I}\begin{pmatrix} m_{12}+m_{21} \\ \mathbf{m}_1 \end{pmatrix}. \end{aligned} \tag{4.3.7}$$

Now from (4.3.2f) we obtain that

$$\begin{pmatrix} m_{11} \\ \mathbf{m}_1 \end{pmatrix}=\begin{pmatrix} D_x^{-1} & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} m_{12}+m_{21} \\ \mathbf{m}_1 \end{pmatrix}. \tag{4.3.8}$$

The last step is to substitute the equations (4.3.7) and (4.3.8) in the evolution (4.3.5) to prove the statement of the theorem:

$$\begin{aligned} \begin{pmatrix} D_t u \\ D_t\mathbf{u} \end{pmatrix} &= \mathfrak{H}\mathfrak{I}\begin{pmatrix} m_{12}+m_{21} \\ \mathbf{m}_1 \end{pmatrix} \\ &\quad +\begin{pmatrix} D_x-C_u & 0 \\ -L_{\mathbf{u}} & 0 \end{pmatrix}\begin{pmatrix} D_x^{-1} & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} m_{12}+m_{21} \\ \mathbf{m}_1 \end{pmatrix}+\begin{pmatrix} m_{12}+m_{21} \\ \mathbf{m}_1 \end{pmatrix} \\ &= \mathfrak{H}\mathfrak{I}\begin{pmatrix} m_{12}+m_{21} \\ \mathbf{m}_1 \end{pmatrix} \\ &\quad +\left(\begin{pmatrix} (D_x-C_u)D_x^{-1} & 0 \\ -L_{\mathbf{u}}D_x^{-1} & 0 \end{pmatrix}+\begin{pmatrix} D_xD_x^{-1} & 0 \\ 0 & I \end{pmatrix}\right)\begin{pmatrix} m_{12}+m_{21} \\ \mathbf{m}_1 \end{pmatrix} \\ &= \mathfrak{H}\mathfrak{I}\begin{pmatrix} m_{12}+m_{21} \\ \mathbf{m}_1 \end{pmatrix}+\mathfrak{A}\begin{pmatrix} m_{12}+m_{21} \\ \mathbf{m}_1 \end{pmatrix}. \end{aligned}$$

■

Remark 4.3.4. Similar to Remark 4.2.4 in the Riemannian geometry of signature p, q, if we apply the Killing form formula (1.4.1) in symplectic geometry, then we can see that

$$\begin{aligned} m_{21} - m_{21} &= D_x^{-1}(\frac{1}{2}A_u(m_{12}+m_{21}) - A_{\mathbf{u}}\mathbf{m}_1) \\ &= D_x^{-1}K(\begin{pmatrix} 0 & 0 & 0 \\ 0 & u & -\overline{\mathbf{u}}^t \\ 0 & \mathbf{u} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & m_{12}+m_{21} & -\overline{\mathbf{m}}_1^t \\ 0 & \mathbf{m}_1 & 0 \end{pmatrix}). \end{aligned}$$

Of course up to some constant coefficient.

Let us put $\begin{pmatrix} h \\ \mathbf{h} \end{pmatrix} = \mathfrak{A}\begin{pmatrix} m_{12}+m_{21} \\ \mathbf{m}_1 \end{pmatrix}$. Then we obtain

$$\begin{pmatrix} m_{12}+m_{21} \\ \mathbf{m}_1 \end{pmatrix} = \mathfrak{A}^{-1}\begin{pmatrix} h \\ \mathbf{h} \end{pmatrix}, \quad \mathfrak{A}^{-1} = \begin{pmatrix} D_x(2D_x - C_u)^{-1} & 0 \\ L_{\mathbf{u}}(2D_x - C_u)^{-1} & I \end{pmatrix}.$$

Hence the evolution in the theorem takes the following form:

$$\begin{pmatrix} D_t u \\ D_t \mathbf{u} \end{pmatrix} = \mathfrak{R}\begin{pmatrix} h \\ \mathbf{h} \end{pmatrix} + \begin{pmatrix} h \\ \mathbf{h} \end{pmatrix}, \quad \mathfrak{R} = \mathfrak{H}\mathfrak{I}\mathfrak{A}^{-1}. \tag{4.3.9}$$

If we make the specialization $\begin{pmatrix} h \\ \mathbf{h} \end{pmatrix} = \begin{pmatrix} u_1 \\ \mathbf{u}_1 \end{pmatrix}$, where $u_1, \mathbf{u}_1$ are the derivatives of u and $\mathbf{u}$ with respect to x, respectively, then we obtain the noncommutative evolution equations:

$$\left\{\begin{aligned} u_t &= \frac{1}{4}u_3 + \frac{3}{8}(-uu_1u - uu_2 + u_2u) + \frac{3}{2}\langle\mathbf{u},\mathbf{u}\rangle u_1 + \langle\mathbf{u},\mathbf{u}_1\rangle u + \tfrac{1}{2}u\langle\mathbf{u},\mathbf{u}_1\rangle \\ &\quad +2u\langle\mathbf{u}_1,\mathbf{u}\rangle - \tfrac{1}{2}\langle\mathbf{u}_1,\mathbf{u}\rangle u + \frac{3}{2}C_{\mathbf{u}}\mathbf{u}_2, \\ \mathbf{u}_t &= \mathbf{u}_3 + \frac{3}{2}\mathbf{u}_2 u + \frac{3}{4}\mathbf{u}_1(u_1 + \frac{1}{2}u^2 + 2\langle\mathbf{u},\mathbf{u}\rangle). \end{aligned}\right. \tag{4.3.10}$$

Remark 4.3.5. It is remarkable to see how the procedure will go if we choose

$$\omega(D_x) = \begin{pmatrix} \sigma & 1 & \mathbf{0}^t \\ -1 & u & -\overline{\mathbf{u}}^t \\ \mathbf{0} & \mathbf{u} & \mathbf{0} \end{pmatrix}, \quad \omega(D_t) = \begin{pmatrix} m_{11} & m_{12} & -\overline{\mathbf{m}}_1^t \\ m_{21} & m_{22} & -\overline{\mathbf{m}}_2^t \\ \mathbf{m}_1 & \mathbf{m}_2 & \mathbf{M} \end{pmatrix}.$$

In this case, the Cartan structure equation on the manifold with zero constant curvature leads to the following evolution:

$$\begin{aligned} \begin{pmatrix} D_t\sigma \\ D_t u \\ D_t\mathbf{u} \end{pmatrix} &= \begin{pmatrix} D_x - C_\sigma & 0 & 0 \\ 0 & D - C_u & C_{\mathbf{u}} \\ 0 & -L_{\mathbf{u}} & D_x + \mathfrak{H} + \hat{R}_u \end{pmatrix}\begin{pmatrix} m_{11} - m_{22} \\ m_{22} - m_{11} \\ \mathbf{m}_2 \end{pmatrix} \\ &\quad + \begin{pmatrix} (D - C_\sigma)m_{22} \\ (D - C_u)m_{11} \\ -L_{\mathbf{u}}m_{11} \end{pmatrix} + \begin{pmatrix} -m_{12} - m_{21} \\ m_{12} + m_{21} \\ \mathbf{m}_1 \end{pmatrix}. \end{aligned}$$

Then, with little effort, we realize that we can not solve these equations. Therefore the dimension of the gauge matrix is a criterium for the Cartan structure to be solved. The problem remains why if we put u in entry $(2,2)$ of the gauge matrix instead of the entry $(1,1)$ of this matrix, we succeed to find all the integrable properties, but not in entry $(1,1)$. In other words, the choice of the frame work is so important that if we change it even very slightly, we will get either nasty extra nonlocal expression which we can not solve or trivial equations. Notice that in this sense, facing such problems, the situation with the Lax method is the same.

Now we give the specific version of Definition 2.2.6 for the pairing in the current geometry.

Definition 4.3.1. The *pairing* between $\begin{pmatrix} h \\ \mathbf{h} \end{pmatrix}$ and $\begin{pmatrix} g \\ \mathbf{g} \end{pmatrix}$ is defined by

$$\langle \begin{pmatrix} h \\ \mathbf{h} \end{pmatrix}, \begin{pmatrix} g \\ \mathbf{g} \end{pmatrix} \rangle = \int K(\sigma \begin{pmatrix} h \\ \mathbf{h} \end{pmatrix}, \sigma \begin{pmatrix} g \\ \mathbf{g} \end{pmatrix}),$$

in which σ is a section of $\mathfrak{g}/\mathfrak{h}$ subject to the zero constant curvature condition into the subspace of the Lie algebra $\mathfrak{g}$ generated just as the Cartan matrix $\omega(D_x)$. For instance, one can take

$$\sigma \begin{pmatrix} h \\ \mathbf{h} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & h & -\overline{\mathbf{h}}^T \\ 0 & \mathbf{h} & 0 \end{pmatrix}.$$

The adjoint of the operator P has been defined in (2.2.8). In the current geometry, this definition reduces to:

$$\langle \begin{pmatrix} h \\ \mathbf{h} \end{pmatrix}, P \begin{pmatrix} g \\ \mathbf{g} \end{pmatrix} \rangle = \langle P^* \begin{pmatrix} h \\ \mathbf{h} \end{pmatrix}, \begin{pmatrix} g \\ \mathbf{g} \end{pmatrix} \rangle.$$

Since the pairing is nondegenerate, P^* is well-defined.

Example 4.3.1. We compute the adjoint operator of the operator $\mathfrak{A}$. We see that $\mathfrak{A}^* = \begin{pmatrix} D_x^{-1}(2D_x - C_u) & D_x^{-1}C_{\mathbf{u}} \\ 0 & I \end{pmatrix}$. This can be done in a few steps. Let us put $\mathfrak{A}_1 = \begin{pmatrix} C_u & 0 \\ 0 & 0 \end{pmatrix}$. Then we compute its adjoint as follows.

$$\begin{aligned}
\langle \begin{pmatrix} h \\ \mathbf{h} \end{pmatrix}, \mathfrak{A}_1 \begin{pmatrix} g \\ \mathbf{g} \end{pmatrix} \rangle &= \int K(\begin{pmatrix} 0 & 0 & 0 \\ 0 & h & -\overline{\mathbf{h}}^T \\ 0 & \mathbf{h} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & C_u g & 0 \\ 0 & 0 & 0 \end{pmatrix}) \\
&= \int K(\begin{pmatrix} h & -\overline{\mathbf{h}}^T \\ \mathbf{h} & 0 \end{pmatrix}, \begin{pmatrix} C_u g & 0 \\ 0 & 0 \end{pmatrix}) = \int K(\begin{pmatrix} h & -\overline{\mathbf{h}}^T \\ \mathbf{h} & 0 \end{pmatrix}, \left[\begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix} \right]) \\
&= -\int K(\begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix}, \left[\begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} h & -\overline{\mathbf{h}}^T \\ \mathbf{h} & 0 \end{pmatrix} \right]) = -\int K(\begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} C_u h & 0 \\ 0 & 0 \end{pmatrix}) \\
&= -\int K(\begin{pmatrix} g & -\overline{\mathbf{g}}^T \\ \mathbf{g} & 0 \end{pmatrix}, \begin{pmatrix} C_u h & 0 \\ 0 & 0 \end{pmatrix}) = -\langle \begin{pmatrix} g \\ \mathbf{g} \end{pmatrix}, \mathfrak{A}_1 \begin{pmatrix} h \\ \mathbf{h} \end{pmatrix} \rangle,
\end{aligned}$$

in which we have used Lemmas 1.4.2 and 1.4.3 for some of the equalities. Thus $\mathfrak{A}_1^* = -\mathfrak{A}_1$. Now let us put $\mathfrak{A}_2 = \begin{pmatrix} 0 & 0 \\ L_{\mathbf{u}} & 0 \end{pmatrix}$. Hence we have that

$$\begin{aligned}
\langle \begin{pmatrix} h \\ \mathbf{h} \end{pmatrix}, \mathfrak{A}_1 \begin{pmatrix} g \\ \mathbf{g} \end{pmatrix} \rangle &= \int K(\begin{pmatrix} 0 & 0 & 0 \\ 0 & h & -\overline{\mathbf{h}}^T \\ 0 & \mathbf{h} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\overline{L_{\mathbf{u}} g}^t \\ 0 & L_{\mathbf{u}} g & 0 \end{pmatrix}) \\
&= \int K(\begin{pmatrix} h & -\overline{\mathbf{h}}^T \\ \mathbf{h} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\overline{L_{\mathbf{u}} g}^t \\ L_{\mathbf{u}} g & 0 \end{pmatrix}) \\
&= \int K(\begin{pmatrix} h & -\overline{\mathbf{h}}^T \\ \mathbf{h} & 0 \end{pmatrix}, \left[\begin{pmatrix} 0 & -\overline{\mathbf{u}}^T \\ \mathbf{u} & 0 \end{pmatrix}, \begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix} \right]) \\
&= -\int K(\begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix}, \left[\begin{pmatrix} 0 & -\overline{\mathbf{u}}^T \\ \mathbf{u} & 0 \end{pmatrix}, \begin{pmatrix} h & -\overline{\mathbf{h}}^T \\ \mathbf{h} & 0 \end{pmatrix} \right]) \\
&= -\int K(\begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -C_{\mathbf{u}} \mathbf{h} & -\overline{\mathbf{u} h}^T \\ \mathbf{u} h & \mathbf{h} \overline{\mathbf{u}}^t - \mathbf{u} \overline{\mathbf{h}}^t \end{pmatrix}) \\
&= -\int K(\begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -C_{\mathbf{u}} \mathbf{h} & 0 \\ 0 & 0 \end{pmatrix}) \\
&= \int K(\begin{pmatrix} g & -\overline{\mathbf{g}}^t \\ \mathbf{g} & 0 \end{pmatrix}, \begin{pmatrix} C_{\mathbf{u}} \mathbf{h} & 0 \\ 0 & 0 \end{pmatrix}) \\
&= \langle \begin{pmatrix} g \\ \mathbf{g} \end{pmatrix}, \mathfrak{A}_2^* \begin{pmatrix} h \\ \mathbf{h} \end{pmatrix} \rangle,
\end{aligned}$$

where $\mathfrak{A}_2^* = \begin{pmatrix} 0 & C_{\mathbf{u}} \\ 0 & 0 \end{pmatrix}$. Here again we have used the Lemmas 1.4.2 and 1.4.3 and the fact that the Killing form is invariant under adjoint action. Now it is clear that $(D_x^{-1})^* = -D_x^{-1}$. Thus we have that

$$\begin{pmatrix} C_u D_x^{-1} & 0 \\ 0 & 0 \end{pmatrix}^* = (\mathfrak{A}_1 D_x^{-1})^* = (D_x^{-1})^* \mathfrak{A}_1^* = D_x^{-1} \mathfrak{A}_1 = \begin{pmatrix} D_x^{-1} C_u & 0 \\ 0 & 0 \end{pmatrix}.$$

Similarly we do have that

$$\begin{pmatrix} 0 & 0 \\ L_{\mathbf{u}} D_x^{-1} & 0 \end{pmatrix}^* = (\mathfrak{A}_2 D_x^{-1})^* = (D_x^{-1})^* \mathfrak{A}_2^* = \begin{pmatrix} 0 & -D_x^{-1} C_{\mathbf{u}} \\ 0 & 0 \end{pmatrix}.$$

Hence it should be clear that

$$\mathfrak{A}^* = \begin{pmatrix} D_x^{-1}(2D_x - C_u) & D_x^{-1} C_{\mathbf{u}} \\ 0 & I \end{pmatrix}.$$

In the following proposition we show that there is a meaningful link between the operator $\mathfrak{H}$ and $\mathfrak{A}$.

Proposition 4.3.6. The following equality holds

$$\mathfrak{A}\mathfrak{H} = \mathfrak{H}\mathfrak{A}^*.$$

Proof. It is simple multiplication of two matrix operator as follows.

$$\begin{aligned}
\mathfrak{A}\mathfrak{H} - \mathfrak{H}\mathfrak{A}^* &= \begin{pmatrix} (2D_x - C_u)D_x^{-1} & 0 \\ -L_{\mathbf{u}}D_x^{-1} & I \end{pmatrix} \begin{pmatrix} D_x - C_u & C_{\mathbf{u}} \\ -L_{\mathbf{u}} & D_x + R_u + \mathfrak{H}_1 \end{pmatrix} \\
&\quad - \begin{pmatrix} D_x - C_u & C_{\mathbf{u}} \\ -L_{\mathbf{u}} & D_x + R_u + \mathfrak{H}_1 \end{pmatrix} \begin{pmatrix} D_x^{-1}(2D_x - C_u) & D_x^{-1}C_{\mathbf{u}} \\ 0 & I \end{pmatrix} \\
&= \begin{pmatrix} 2D_x - 3C_u + C_uD_x^{-1}C_u & (2D_x - C_u)D_x^{-1}C_{\mathbf{u}} \\ -2L_{\mathbf{u}} + L_{\mathbf{u}}D_x^{-1}C_u & -L_{\mathbf{u}}D_x^{-1}C_{\mathbf{u}} + D_x + R_u + \mathfrak{H}_1 \end{pmatrix} \\
&\quad - \begin{pmatrix} 2D_x - 3C_u + C_uD_x^{-1}C_u & 2C_{\mathbf{u}} - C_uD_x^{-1}C_{\mathbf{u}} \\ -2L_{\mathbf{u}} + L_{\mathbf{u}}D_x^{-1}C_u & -L_{\mathbf{u}}D_x^{-1}C_{\mathbf{u}} + D_x + R_u + \mathfrak{H}_1 \end{pmatrix} \\
&= 0
\end{aligned}$$

■

Corollary 4.3.7. The operator $\mathfrak{H}$ is antisymmetric as well as $\mathfrak{H}\mathfrak{A}^*$, that is $\mathfrak{H}^* = -\mathfrak{H}$ and $(\mathfrak{H}\mathfrak{A}^*)^* = -\mathfrak{H}\mathfrak{A}^*$. Furthermore the operator $\mathfrak{I}$ itself is also antisymmetric.

Proof. We decompose the operator $\mathfrak{H}$ into two operators and seperately prove that each of them is antisymmetric. Let us denote the operator $\begin{pmatrix} 0 & 0 \\ 0 & R_u \end{pmatrix}$ by P_1. Then we can compute its adjoint as follows:

$$\begin{aligned}
&\langle \begin{pmatrix} h \\ \mathbf{h} \end{pmatrix}, P_1 \begin{pmatrix} g \\ \mathbf{g} \end{pmatrix} \rangle = \int K(\begin{pmatrix} h & -\overline{\mathbf{h}}^T \\ \mathbf{h} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\overline{\mathbf{g}u}^T \\ \mathbf{g}u & 0 \end{pmatrix}) \\
= &\int K(\begin{pmatrix} h & -\overline{\mathbf{h}}^T \\ \mathbf{h} & 0 \end{pmatrix}, \left[\begin{pmatrix} 0 & -\overline{\mathbf{g}}^T \\ \mathbf{g} & 0 \end{pmatrix}, \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} \right]) \\
= &\int K(\begin{pmatrix} 0 & -\overline{\mathbf{g}}^T \\ \mathbf{g} & 0 \end{pmatrix}, \left[\begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} h & -\overline{\mathbf{h}}^T \\ \mathbf{h} & 0 \end{pmatrix} \right]) \\
= &\int K(\begin{pmatrix} 0 & -\overline{\mathbf{g}}^T \\ \mathbf{g} & 0 \end{pmatrix}, \begin{pmatrix} C_uh & \overline{\mathbf{h}u}^t \\ -\mathbf{h}u & 0 \end{pmatrix}) = \int K(\begin{pmatrix} 0 & -\overline{\mathbf{g}}^T \\ \mathbf{g} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \overline{\mathbf{h}u}^t \\ -\mathbf{h}u & 0 \end{pmatrix}) \\
= &\int K(\begin{pmatrix} g & -\overline{\mathbf{g}}^T \\ \mathbf{g} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \overline{\mathbf{h}u}^t \\ -\mathbf{h}u & 0 \end{pmatrix}) = \langle \begin{pmatrix} g \\ \mathbf{g} \end{pmatrix}, -P_1 \begin{pmatrix} h \\ \mathbf{h} \end{pmatrix} \rangle.
\end{aligned}$$

This indeed shows that $P_1^* = -P_1$. Now put $P_2 = \begin{pmatrix} 0 & 0 \\ 0 & \mathfrak{H}_1 \end{pmatrix}$. Then using the notations defined in section 4.4 and Lemmas 4.4.1, we see that

$$
\begin{aligned}
&\langle \begin{pmatrix} h \\ \mathbf{h} \end{pmatrix}, P_2 \begin{pmatrix} g \\ \mathbf{g} \end{pmatrix} \rangle \\
= \;& \int K(\begin{pmatrix} 0 & 0 & 0 \\ 0 & h & -\overline{\mathbf{h}}^t \\ 0 & \mathbf{h} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\overline{D_x^{-1}(\mathbf{g}\overline{\mathbf{u}}^t - \mathbf{u}\overline{\mathbf{g}}^t)\mathbf{u}}^t \\ 0 & D_x^{-1}(\mathbf{g}\overline{\mathbf{u}}^t - \mathbf{u}\overline{\mathbf{g}}^t)\mathbf{u} & 0 \end{pmatrix}) \\
= \;& -\int K(\begin{pmatrix} 0 & 0 & 0 \\ 0 & h & -\overline{\mathbf{h}}^t \\ 0 & \mathbf{h} & 0 \end{pmatrix}, \\
& \Big[\begin{pmatrix} 0 & 0 & 0 \\ 0 & u & -\bar{\mathbf{u}}^t \\ 0 & \mathbf{u} & 0 \end{pmatrix}, D_x^{-1}\pi_0 \Big[\begin{pmatrix} 0 & 0 & 0 \\ 0 & u & -\bar{\mathbf{u}}^t \\ 0 & \mathbf{u} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & g & -\bar{\mathbf{g}}^t \\ 0 & \mathbf{g} & 0 \end{pmatrix} \Big]\Big]) \\
= \;& \int K(\Big[\begin{pmatrix} 0 & 0 & 0 \\ 0 & u & -\bar{\mathbf{u}}^t \\ 0 & \mathbf{u} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & h & -\overline{\mathbf{h}}^t \\ 0 & \mathbf{h} & 0 \end{pmatrix} \Big], \\
& D_x^{-1}\pi_0 \Big[\begin{pmatrix} 0 & 0 & 0 \\ 0 & u & -\bar{\mathbf{u}}^t \\ 0 & \mathbf{u} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & g & -\bar{\mathbf{g}}^t \\ 0 & \mathbf{g} & 0 \end{pmatrix} \Big]) \\
= \;& -\int K(D_x^{-1}\Big[\begin{pmatrix} 0 & 0 & 0 \\ 0 & u & -\bar{\mathbf{u}}^t \\ 0 & \mathbf{u} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & h & -\overline{\mathbf{h}}^t \\ 0 & \mathbf{h} & 0 \end{pmatrix} \Big], \\
& \pi_0 \Big[\begin{pmatrix} 0 & 0 & 0 \\ 0 & u & -\bar{\mathbf{u}}^t \\ 0 & \mathbf{u} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & g & -\bar{\mathbf{g}}^t \\ 0 & \mathbf{g} & 0 \end{pmatrix} \Big]) \\
= \;& -\int K(D_x^{-1}\pi_0\Big[\begin{pmatrix} 0 & 0 & 0 \\ 0 & u & -\bar{\mathbf{u}}^t \\ 0 & \mathbf{u} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & h & -\overline{\mathbf{h}}^t \\ 0 & \mathbf{h} & 0 \end{pmatrix} \Big], \\
& \Big[\begin{pmatrix} 0 & 0 & 0 \\ 0 & u & -\bar{\mathbf{u}}^t \\ 0 & \mathbf{u} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & g & -\bar{\mathbf{g}}^t \\ 0 & \mathbf{g} & 0 \end{pmatrix} \Big]) \\
= \;& -\int K(\Big[\begin{pmatrix} 0 & 0 & 0 \\ 0 & u & -\bar{\mathbf{u}}^t \\ 0 & \mathbf{u} & 0 \end{pmatrix}, D_x^{-1}\pi_0 \Big[\begin{pmatrix} 0 & 0 & 0 \\ 0 & u & -\bar{\mathbf{u}}^t \\ 0 & \mathbf{u} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & h & -\overline{\mathbf{h}}^t \\ 0 & \mathbf{h} & 0 \end{pmatrix} \Big]\Big], \\
& \begin{pmatrix} 0 & 0 & 0 \\ 0 & g & -\bar{\mathbf{g}}^t \\ 0 & \mathbf{g} & 0 \end{pmatrix}) \\
= \;& -\int K(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\overline{D_x^{-1}(\mathbf{h}\overline{\mathbf{u}}^t - \mathbf{u}\overline{\mathbf{h}}^t)\mathbf{u}}^t \\ 0 & D_x^{-1}(\mathbf{h}\overline{\mathbf{u}}^t - \mathbf{u}\overline{\mathbf{h}}^t)\mathbf{u} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & g & -\bar{\mathbf{g}}^t \\ 0 & \mathbf{g} & 0 \end{pmatrix}).
\end{aligned}
$$

The last expression is nothing but $\langle \begin{pmatrix} g \\ \mathbf{g} \end{pmatrix}, -P_2 \begin{pmatrix} h \\ \mathbf{h} \end{pmatrix} \rangle$. Hence $P_2^* = -P_2$. Using the computation in example 4.3.1 together with the two last identities shows that the operator $\mathfrak{H}$ is indeed antisymmetric.

In order to prove that the operator $\mathfrak{I}$ is antisymmetric, having proved part of that, let us first denote $P_3 = \begin{pmatrix} 0 & 0 \\ 0 & L_{\mathbf{u}} D_x^{-1} A_{\mathbf{u}} \end{pmatrix}$. Then we can compute its adjoint as follows:

$$\begin{aligned}
\langle \begin{pmatrix} h \\ \mathbf{h} \end{pmatrix}, P_2 \begin{pmatrix} g \\ \mathbf{g} \end{pmatrix} \rangle &= \int K(\begin{pmatrix} h & -\overline{\mathbf{h}}^t \\ \mathbf{h} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\overline{L_{\mathbf{u}} D_x^{-1} A_{\mathbf{u}} \mathbf{g}}^t \\ L_{\mathbf{u}} D_x^{-1} A_{\mathbf{u}} \mathbf{g} & 0 \end{pmatrix}) \\
&= \int K(\begin{pmatrix} h & -\overline{\mathbf{h}}^t \\ \mathbf{h} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\overline{\mathbf{u}}^t D_x^{-1} A_{\mathbf{u}} \mathbf{g} \\ \mathbf{u} D_x^{-1} A_{\mathbf{u}} \mathbf{g} & 0 \end{pmatrix}) \\
&= \int K(\begin{pmatrix} h & -\overline{\mathbf{h}}^t \\ \mathbf{h} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\overline{\mathbf{u}}^t \\ \mathbf{u} & 0 \end{pmatrix}).D_x^{-1} A_{\mathbf{u}} \mathbf{g} \\
&= -\int D_x^{-1} K(\begin{pmatrix} h & -\overline{\mathbf{h}}^t \\ \mathbf{h} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\overline{\mathbf{u}}^t \\ \mathbf{u} & 0 \end{pmatrix}).A_{\mathbf{u}} \mathbf{g} \\
&= \int (D_x^{-1} A_{\mathbf{u}} \mathbf{h}).(A_{\mathbf{u}} \mathbf{g}) \\
&= -\int (D_x^{-1} A_{\mathbf{u}} \mathbf{h}).K(\begin{pmatrix} 0 & -\overline{\mathbf{u}}^t \\ \mathbf{u} & 0 \end{pmatrix}, \begin{pmatrix} g & -\overline{\mathbf{g}}^t \\ \mathbf{g} & 0 \end{pmatrix}) \\
&= -\int K(\begin{pmatrix} 0 & -\overline{\mathbf{u}}^t D_x^{-1} A_{\mathbf{u}} \mathbf{h} \\ \mathbf{u} D_x^{-1} A_{\mathbf{u}} \mathbf{h} & 0 \end{pmatrix}, \begin{pmatrix} g & -\overline{\mathbf{g}}^t \\ \mathbf{g} & 0 \end{pmatrix}) \\
&= \langle \begin{pmatrix} g \\ \mathbf{g} \end{pmatrix}, -P_2 \begin{pmatrix} h \\ \mathbf{h} \end{pmatrix} \rangle.
\end{aligned}$$

Therefore $P_2^* = -P_2$. Now it should be clear that $\mathfrak{I}$ is antisymmetric. ∎

Theorem 4.3.8. The operator $N = \mathfrak{A}$ is indeed a Nijenhuis operator. That is, the Nijenhuis tensor vanishes.

Proof. According to Definition 2.5.1, we have to prove that the identity (4.3.11) holds:

$$D_N[N\psi](\varphi) - D_N[N\varphi](\psi) + N(D_N[\varphi](\psi) - D_N[\psi](\varphi)) = 0 \tag{4.3.11}$$

for any pair of vector fields $\varphi, \psi \in \mathfrak{X}(M)$. In our specific case, the φ and ψ take the following form: $\varphi = \begin{pmatrix} p \\ \mathbf{p} \end{pmatrix}, \psi = \begin{pmatrix} q \\ \mathbf{q} \end{pmatrix}$.

Here we are given $N\varphi = \begin{pmatrix} (2D_x - C_u) D_x^{-1} p \\ -L_{\mathbf{u}} D_x^{-1} p + \mathbf{p} \end{pmatrix}$. Now we compute the Fréchet derivative of N as follows and use it later on with different arguments

$$D_N[\psi](\varphi) = \begin{pmatrix} -C_q D_x^{-1} & 0 \\ -L_{\mathbf{q}} D_x^{-1} & 0 \end{pmatrix} \begin{pmatrix} p \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} -C_q D_x^{-1} p \\ -L_{\mathbf{q}} D_x^{-1} p \end{pmatrix}.$$

Now we see that

$$\begin{aligned}
& D_N[N\psi](\varphi) - D_N[N\varphi](\psi) \\
= & \begin{pmatrix} -C_{(2D_x-C_u)D_x^{-1}q} D_x^{-1}p \\ -L_{-L_{\mathbf{u}}D_x^{-1}q+\mathbf{q}} D_x^{-1}p \end{pmatrix} - \begin{pmatrix} -C_{(2D_x-C_u)D_x^{-1}p} D_x^{-1}q \\ -L_{-L_{\mathbf{u}}D_x^{-1}p+\mathbf{p}} D_x^{-1}q \end{pmatrix} \\
= & \begin{pmatrix} -C_{(2D_x-C_u)D_x^{-1}q} D_x^{-1}p + C_{(2D_x-C_u)D_x^{-1}p} D_x^{-1}q \\ -L_{-L_{\mathbf{u}}D_x^{-1}q+\mathbf{q}} D_x^{-1}p + L_{-L_{\mathbf{u}}D_x^{-1}p+\mathbf{p}} D_x^{-1}q \end{pmatrix}.
\end{aligned}$$

On the other hand we obtain

$$\begin{aligned}
D_N[\varphi](\psi) - D_N[\psi](\varphi) &= \begin{pmatrix} -C_p D_x^{-1}q \\ -L_{\mathbf{p}} D_x^{-1}q \end{pmatrix} - \begin{pmatrix} -C_q D_x^{-1}p \\ -L_{\mathbf{q}} D_x^{-1}p \end{pmatrix} \\
&= \begin{pmatrix} -C_p D_x^{-1}q + C_q D_x^{-1}p \\ -L_{\mathbf{p}} D_x^{-1}q + L_{\mathbf{q}} D_x^{-1}p \end{pmatrix}.
\end{aligned}$$

Hence we get that

$$\begin{aligned}
& N\Big(D_N[\varphi](\psi) - D_N[\psi](\varphi)\Big) \\
= & \begin{pmatrix} (2D_x - C_u)D_x^{-1}\Big(- C_p D_x^{-1}q + C_q D_x^{-1}p\Big) \\ -L_{\mathbf{u}} D_x^{-1}\Big(- C_p D_x^{-1}q + C_q D_x^{-1}p\Big) + (-L_{\mathbf{p}} D_x^{-1}q + L_{\mathbf{q}} D_x^{-1}p) \end{pmatrix}
\end{aligned}$$

Now first we prove that the scalar part or the first component of expression

$$D_N[N\psi](\varphi) - D_N[N\varphi](\psi) + N(D_N[\varphi](\psi) - D_N[\psi](\varphi))$$

in (4.3.11) vanishes:

$$\begin{aligned}
& -C_{(2D_x-C_u)D_x^{-1}q} D_x^{-1}p + C_{(2D_x-C_u)D_x^{-1}p} D_x^{-1}q \\
& +(2D_x - C_u)D_x^{-1}\Big(- C_p D_x^{-1}q + C_q D_x^{-1}p\Big) \\
= \; & -C_{(2D_x-C_u)D_x^{-1}q} D_x^{-1}p + C_{(2D_x-C_u)D_x^{-1}p} D_x^{-1}q \\
& +2(-C_p D_x^{-1}q + C_q D_x^{-1}p) - C_u D_x^{-1}\Big(- C_p D_x^{-1}q + C_q D_x^{-1}p\Big) \\
= \; & -2C_q D_x^{-1}p + C_{C_u D_x^{-1}q} D_x^{-1}p + 2C_p D_x^{-1}q - C_{C_u D_x^{-1}p} D_x^{-1}q \\
& +2(-C_p D_x^{-1}q + C_q D_x^{-1}p) - C_u D_x^{-1}\Big(- C_p D_x^{-1}q + C_q D_x^{-1}p\Big) \\
= \; & +C_{C_u D_x^{-1}q} D_x^{-1}p - C_{C_u D_x^{-1}p} D_x^{-1}q - C_u D_x^{-1}\Big(- C_p D_x^{-1}q + C_q D_x^{-1}p\Big) \\
\overset{\text{Jacobi id}}{=} \; & C_u C_{D_x^{-1}q} D_x^{-1}p - C_u D_x^{-1}\Big(- C_p D_x^{-1}q + C_q D_x^{-1}p\Big) \\
= \; & C_u D_x^{-1} D_x C_{D_x^{-1}q} D_x^{-1}p - C_u D_x^{-1}\Big(- C_p D_x^{-1}q + C_q D_x^{-1}p\Big) \\
= \; & C_u D_x^{-1}\Big(C_q D_x^{-1}p - C_p D_x^{-1}q\Big) - C_u D_x^{-1}\Big(- C_p D_x^{-1}q + C_q D_x^{-1}p\Big) \\
= \; & 0,
\end{aligned}$$

We have used once the Jacobi identity as

$$+C_{C_uD_x^{-1}q}D_x^{-1}p - C_{C_uD_x^{-1}p}D_x^{-1}q = C_uC_{D_x^{-1}q}D_x^{-1}p.$$

We manipulate the second component (vector part) as follows:

$$\begin{aligned}
& -L_{-L_{\mathbf{u}}D_x^{-1}q+\mathbf{q}}D_x^{-1}p + L_{-L_{\mathbf{u}}D_x^{-1}p+\mathbf{p}}D_x^{-1}q \\
& -L_{\mathbf{u}}D_x^{-1}\Big(-C_pD_x^{-1}q + C_qD_x^{-1}p\Big) + (-L_{\mathbf{p}}D_x^{-1}q + L_{\mathbf{q}}D_x^{-1}p) \\
= \;& -L_{-L_{\mathbf{u}}D_x^{-1}q}D_x^{-1}p + L_{-L_{\mathbf{u}}D_x^{-1}p}D_x^{-1}q - L_{\mathbf{u}}D_x^{-1}\Big(-C_pD_x^{-1}q + C_qD_x^{-1}p\Big) \\
= \;& L_{\mathbf{u}}(C_{D_x^{-1}q}D_x^{-1}p) - L_{\mathbf{u}}D_x^{-1}\Big(-C_pD_x^{-1}q + C_qD_x^{-1}p\Big) \\
= \;& L_{\mathbf{u}}D_x^{-1}D_x(C_{D_x^{-1}q}D_x^{-1}p) - L_{\mathbf{u}}D_x^{-1}\Big(-C_pD_x^{-1}q + C_qD_x^{-1}p\Big) \\
= \;& L_{\mathbf{u}}D_x^{-1}\Big(C_qD_x^{-1}p - C_pD_x^{-1}q\Big) - L_{\mathbf{u}}D_x^{-1}\Big(-C_pD_x^{-1}q + C_qD_x^{-1}p\Big) \\
= \;& 0,
\end{aligned}$$

In which again we have used the Jacobi identity this time as follows:

$$-L_{-L_{\mathbf{u}}D_x^{-1}q}D_x^{-1}p + L_{-L_{\mathbf{u}}D_x^{-1}p}D_x^{-1}q = L_{\mathbf{u}}(C_{D_x^{-1}q}D_x^{-1}p).$$

This is nothing but the Jacobi identity for the following three elements of $\mathfrak{sp}(n+2)$:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\overline{\mathbf{u}}^t \\ 0 & \mathbf{u} & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & D_x^{-1}p & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & D_x^{-1}q & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

■

Remark 4.3.9. Notice that operator $\mathfrak{A}$ is not recursion operator, consequently it is not invariant under the flow.

Remark 4.3.10. Such an operator $\mathfrak{A}$ appeared as "Starting operator" in Fokas-Santini's papers [60] and [21] where they give the recursion operator and bi-Hamiltonian structure in multidimensional equations.

Remark 4.3.11. Since $N = \mathfrak{A}$ is invertible, so by [14, proposition 3.2], the operator $\mathfrak{A}^{-1}$ itself is Nijenhuis operator.

In the light of Theorem 4.3.8, the operator $\mathfrak{R}$ can be written as

$$\begin{aligned}
\mathfrak{R} &= \mathfrak{H}\mathfrak{I}\mathfrak{A}^{-1} \\
&= (\mathfrak{H}\mathfrak{A}^*)(\mathfrak{A}^{-1*}\mathfrak{I}\mathfrak{A}^{-1}).
\end{aligned}$$

This decomposition of the operator $\mathfrak{R}$ is the key to find Hamiltonian and symplectic operator. In fact the main result of this chapter is the next theorem presenting these facts. In the next section we express these operator in terms of the Lie bracket, Killing form and some projections and in the Chapter 6 we will prove the theorem in detail.

Theorem 4.3.12. The operators $\mathfrak{H}$ and $\mathfrak{H}\mathfrak{A}^*$ are Hamiltonian operators and the operator $\mathfrak{A}^{-1*}\mathfrak{I}\mathfrak{A}^{-1}$ is symplectic.

Remark 4.3.13. Theorems 4.3.8 and 4.3.12 show that the manifold $\mathsf{M} = G/H$ is endowed with a so called 'Poisson-Nijenhuis structure' as is defined in [49]. We will not go further in this direction. For more information, the reader is referred to the works of Magri, as presented in for instance [48] and [39].

Remark 4.3.14. Notice that we can decompose the entry in the first column and first row of $\mathfrak{H}$. One can write

$$\begin{aligned}
& (D_x + R_u)(2D_x - C_u)^{-1}(D_x - L_u) \\
= \; & (D_x + \frac{A_u - C_u}{2})(2D_x - C_u)^{-1}(D_x - \frac{A_u + C_u}{2}) \\
= \; & (\frac{A_u}{2} + \frac{2D_x - C_u}{2})(2D_x - C_u)^{-1}(-\frac{A_u}{2} + \frac{2D_x - C_u}{2}) \\
= \; & \frac{1}{2}D_x - \frac{1}{2}A_u(2D_x - C_u)^{-1}\frac{1}{2}A_u - \frac{1}{4}C_u \\
= \; & \frac{1}{2}D_x - \frac{1}{2}uD_x^{-1}\frac{1}{2}A_u - \frac{1}{4}C_u
\end{aligned}$$

Remember that the operator acts on purely imaginary arguments. Then the last equality follows from $(2D_x - C_u)^{-1} = \frac{1}{2}(D_x - \frac{1}{2}C_u)^{-1}$ and $(D_x - \frac{1}{2}C_u)^{-1}$ acting on real valued function gives D_x^{-1} acting on the same function, as well as A_u acting on the real functions gives twice acting u on the same function. Indeed the result of action A_u on a imaginary valued function is real function. Therefore the operator $(2D_x - C_u)^{-1} = \frac{1}{2}(D_x - \frac{1}{2}C_u)^{-1}$ acting on this real function would yield $\frac{1}{2}D_x^{-1}$ action on the same real function, for assume that f is the real function and $(D_x - \frac{1}{2}C_u)^{-1}f = g$. Then $f = (D_x - \frac{1}{2}C_u)g$. Since $C_u g$ is imaginary, hence we do have that $D_x g_0 = f$ where g_0 is the real part of g. Thus $(D_x - \frac{1}{2}C_u)^{-1}f = g_0$. One can see this from the expansion of $(D_x - \frac{1}{2}C_u)^{-1}$ as well. Indeed $(D_x - \frac{1}{2}C_u)^{-1} = D_x^{-1} + \frac{1}{2}D_x^{-1}C_uD_x^{-1} + ...$ Hence $C_uD_x^{-1}f = 0$. Therefore we have that $(D_x - \frac{1}{2}C_u)^{-1}f = D_x^{-1}f$.

Also notice that A_u acting on the real function f is equal to $2uf$.

This decomposition indicates that if we put $\mathbf{u} = 0$, then geometric operators, Hamiltonian, symplectic and recursion operators, will reduce to the ones that appeared in [31].

4.4 Geometric operators in the form of Lie bracket, Killing form and projections

In the method we are using, the only tools we have are the Lie algebra and the Cartan geometry, hence we expect to be able to write the geometric operators $\mathfrak{H}$ and $\mathfrak{I}$ in terms of the Lie bracket, the Killing form and proper projections. In this sense for instance see Remark 4.3.4. Also as one can see in the proof of Theorem 4.3.8, we could use the

Jacobi identity for the matrices in the Lie algebra $\mathfrak{sp}(n+2)$ which in fact are projections of bigger matrices.

Let us define the projections π_0 and π_1 as follows:

$$\pi_0 \begin{pmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & -\bar{\mathbf{m}}_2^t \\ 0 & \mathbf{m}_2 & \mathbf{M} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{M} \end{pmatrix},$$

and

$$\pi_1 \begin{pmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & -\bar{\mathbf{m}}_2^t \\ 0 & \mathbf{m}_2 & \mathbf{M} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & m_{22} & -\bar{\mathbf{m}}_2^t \\ 0 & \mathbf{m}_2 & 0 \end{pmatrix}.$$

In the following lemma, we give the Lie algebraic form of the operator $\mathfrak{H}_1$ defined as in (4.3.1).

Lemma 4.4.1.

$$\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & u & -\bar{\mathbf{u}}^t \\ 0 & \mathbf{u} & 0 \end{pmatrix}, D_x^{-1}\pi_0\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & u & -\bar{\mathbf{u}}^t \\ 0 & \mathbf{u} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & m_{22} & -\bar{\mathbf{m}}_2^t \\ 0 & \mathbf{m}_2 & 0 \end{pmatrix}\right]\right]$$
$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\bar{\mathbf{u}}^t D^{-1}(-\mathbf{u}\bar{\mathbf{m}}_2^t + \mathbf{m}_2\bar{\mathbf{u}}^t) \\ 0 & -D^{-1}(-\mathbf{u}\bar{\mathbf{m}}_2^t + \mathbf{m}_2\bar{\mathbf{u}}^t)\mathbf{u} & 0 \end{pmatrix}.$$

Proof. The proof just follows from computing the Lie bracket of the elements of the Lie algebra $\mathfrak{sp}(n+2)$. ■

Let $\hat{u}$ and $\hat{m}_2$ be the projection of $\omega(D_x)$ and $\omega(D_t)$ over the Lie subalgebra $\mathfrak{h}$, respectively as well as $\hat{a}$ and $\hat{m}_1$, the projections of $\omega(D_x)$ and $\omega(D_t)$ over the vector space $\mathfrak{g}/\mathfrak{h}$ which itself is indeed the dual orthogonal of $\mathfrak{h}$ with respect to the Killing form. In other words

$$\hat{u} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & u & -\bar{\mathbf{u}}^t \\ 0 & \mathbf{u} & 0 \end{pmatrix}, \quad \hat{a} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$\hat{m}_2 = \begin{pmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & -\bar{\mathbf{m}}_2^t \\ 0 & \mathbf{m}_2 & \mathbf{M} \end{pmatrix}, \quad \hat{m}_1 = \begin{pmatrix} 0 & m_{12} & -\bar{\mathbf{m}}_1^t \\ m_{21} & 0 & 0 \\ \mathbf{m}_1 & 0 & 0 \end{pmatrix}.$$

Now let us define the projections ρ_0 and ρ_1 on the diagonal and offdiagonal of the image of π_1 as follows:

$$\rho_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & m_{22} & -\bar{\mathbf{m}}_2^t \\ 0 & \mathbf{m}_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & m_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\rho_0 \begin{pmatrix} 0 & 0 & 0 \\ 0 & m_{22} & -\bar{\mathbf{m}}_2^t \\ 0 & \mathbf{m}_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\bar{\mathbf{m}}_2^t \\ 0 & \mathbf{m}_2 & 0 \end{pmatrix}.$$

We give some identities which in fact are interactions of projections ρ_0, ρ_1 and π_1.

Proposition 4.4.2. For every matrix $\hat{q}$ in the image of π_1, we have that

1. $(\frac{1}{2}\rho_1 + \rho_0)\pi_1 \mathrm{ad}_{\hat{a}}^2 \hat{q} = -\hat{q}$.
2. $\pi_1 \mathrm{ad}_{\hat{a}} \mathrm{ad}_{\hat{u}} \mathrm{ad}_{\hat{a}} \hat{q} = -\mathrm{ad}_{\hat{u}} \rho_1 \hat{q} - \pi_1 \mathrm{ad}_{\rho_0 \hat{u}} \rho_0 \hat{q}$,

The following theorem describes how we can express the geometric operator in terms of Lie algebraic notions, such as Killing form, adjoint representation and the projections.

Theorem 4.4.3. The evolution of the $\hat{u}$ following the Cartan structure equation on $M = G/H$ can be expressed as

$$\hat{u}_t = \hat{\mathfrak{H}}\hat{\mathfrak{I}}\hat{m}_0 + \hat{\mathfrak{A}}\hat{m}_0,$$

in which the Lie algebra form $\hat{\mathfrak{I}}$ of geometric operator $\mathfrak{I}$ and $\hat{\mathfrak{A}}$ of Nijenhuis operator $\mathfrak{A}$ appears as

$$\begin{aligned} \hat{\mathfrak{H}} &= D_x - \pi_1 \mathrm{ad}_{\hat{u}} - \mathrm{ad}_{\hat{u}} D_x^{-1} \pi_0 \mathrm{ad}_{\hat{u}}, \\ \hat{\mathfrak{I}} &= -\frac{1}{2}\hat{u} D_x^{-1} K(\hat{u}, .) - (\frac{1}{2}\rho_1 + \rho_0)\pi_1 \mathrm{ad}_{\hat{a}} (D_x - \mathrm{ad}_{\hat{u}}) \mathrm{ad}_{\hat{a}} \pi_1 (\frac{1}{2}\rho_1 + \rho_0), \\ \hat{\mathfrak{A}} &= \rho_0 + 2\rho_1 - \mathrm{ad}_{\hat{u}} D_x^{-1} \rho_1. \end{aligned}$$

Proof. From the curvature part or in fact the equations (4.3.2b),(4.3.2c) and (4.3.2d) and the previous lemma, we simply find that

$$\hat{u}_t = \hat{\mathfrak{H}}(\pi_1 \hat{m}_2) + \hat{m}_0, \quad \hat{m}_0 = \pi_1 \mathrm{ad}_{\hat{a}} \hat{m}_1,$$

Now the torsion part gives the following matrix equation:

$$\mathrm{ad}_{\hat{a}}(\hat{m}_2) = (D_x - \mathrm{ad}_{\hat{u}})\hat{m}_1. \tag{4.4.2}$$

Since $\mathrm{ad}_{\hat{a}}^2 \neq \lambda I$ for any $\lambda \in \mathbb{R}$, we can not solve equation (4.4.2) in the usual way. Therefore the existence of the Nijenhuis operator $\mathfrak{A}$ plays a crucial rule in the symplectic geometry. Notice that in the Riemannian case we do have $\mathrm{ad}_{\hat{a}}^2 = -I$.

In order to get rid of this difficulty, we do as follows:

$$\begin{aligned}
\hat{u}_t &= \hat{\mathfrak{H}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & m_{22}-m_{11} & -\bar{\mathbf{m}}_2^t \\ 0 & \mathbf{m}_2 & 0 \end{pmatrix} + \hat{\mathfrak{H}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & m_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \hat{m}_0 \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & m_{22}-m_{11} & -\bar{\mathbf{m}}_2^t \\ 0 & \mathbf{m}_2 & 0 \end{pmatrix} + (D_x - \pi_1 ad_{\hat{u}}) \begin{pmatrix} 0 & 0 & 0 \\ 0 & m_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \hat{m}_0 \\
&= \hat{\mathfrak{H}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & m_{22}-m_{11} & -\bar{\mathbf{m}}_2^t \\ 0 & \mathbf{m}_2 & 0 \end{pmatrix} + \Big((2D_x - \mathrm{ad}_{\hat{u}})D_x^{-1}\rho_1 + \rho_0\Big)\hat{m}_0 \\
&= \hat{\mathfrak{H}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & m_{22}-m_{11} & -\bar{\mathbf{m}}_2^t \\ 0 & \mathbf{m}_2 & 0 \end{pmatrix} + \hat{\mathfrak{A}}\hat{m}_0. \qquad (4.4.3)
\end{aligned}$$

Now the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & m_{22}-m_{11} & -\bar{\mathbf{m}}_2^t \\ 0 & \mathbf{m}_2 & 0 \end{pmatrix}$$

can be expressed in terms of $\hat{m}_2$ and consequencely in terms of $\hat{m}_1$ using the identity (4.4.2) as follows:

$$\begin{aligned}
\begin{pmatrix} 0 & 0 & 0 \\ 0 & m_{22}-m_{11} & -\bar{\mathbf{m}}_2^t \\ 0 & \mathbf{m}_2 & 0 \end{pmatrix} &= -(\frac{1}{2}\rho_1 + \rho_0)\pi_1 \mathrm{ad}_a^2 \hat{m}_2 \\
&= -(\frac{1}{2}\rho_1 + \rho_0)\pi_1 \mathrm{ad}_a (D_x - \mathrm{ad}_{\hat{u}})\hat{m}_1. \qquad (4.4.4)
\end{aligned}$$

Since $m_{21} - m_{12} = D_x^{-1}K(\hat{u}, \hat{m}_0)$, thus the matrix $\hat{m}_1$ can be written in terms of $\hat{m}_0$ as follows.

$$\begin{aligned}
\hat{m}_1 &= \begin{pmatrix} 0 & \dfrac{m_{12}-m_{21}}{2} & 0 \\ \dfrac{m_{21}-m_{12}}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & \dfrac{m_{12}+m_{21}}{2} & -\bar{\mathbf{m}}_1^t \\ \dfrac{m_{21}+m_{12}}{2} & 0 & 0 \\ \mathbf{m}_1 & 0 & 0 \end{pmatrix} \\
&= -\frac{1}{2}\hat{a}D_x^{-1}K(\hat{u}, \hat{m}_0) + \mathrm{ad}_{\hat{a}}(\frac{1}{2}\rho_1 + \rho_0)\hat{m}_0.
\end{aligned}$$

Hence we identify the operator $\hat{\mathfrak{J}}$ as follows:

$$
\begin{aligned}
\begin{pmatrix} 0 & 0 & 0 \\ 0 & m_{22}-m_{11} & -\bar{\mathbf{m}}_2^t \\ 0 & \mathbf{m}_2 & 0 \end{pmatrix} &= -(\frac{1}{2}\rho_1+\rho_0)\pi_1 \mathrm{ad}_{\hat{a}}(D_x-\mathrm{ad}_{\hat{u}})\hat{m}_1 \\
= & -(\frac{1}{2}\rho_1+\rho_0)\pi_1 \mathrm{ad}_{\hat{a}}(D_x-\mathrm{ad}_{\hat{u}})\Big(-\frac{1}{2}\hat{a}D_x^{-1}K(\hat{u},\hat{m}_0)+\mathrm{ad}_{\hat{a}}(\frac{1}{2}\rho_1+\rho_0)\hat{m}_0\Big) \\
= & -\frac{1}{2}(\frac{1}{2}\rho_1+\rho_0)\pi_1 \mathrm{ad}_{\hat{a}}(\mathrm{ad}_{\hat{u}}\hat{a})D_x^{-1}K(\hat{u},\hat{m}_0) \\
& -(\frac{1}{2}\rho_1+\rho_0)\pi_1 \mathrm{ad}_A(D_x-\mathrm{ad}_{\hat{u}})\mathrm{ad}_A(\frac{1}{2}\rho_1+\rho_0)\hat{m}_0 \\
= & +\frac{1}{2}(\frac{1}{2}\rho_1+\rho_0)\pi_1 (\mathrm{ad}_{\hat{a}}^2\hat{u})D_x^{-1}K(\hat{u},\hat{m}_0) \\
& -(\frac{1}{2}\rho_1+\rho_0)\pi_1 \mathrm{ad}_A(D_x-\mathrm{ad}_{\hat{u}})ad_A(\frac{1}{2}\rho_1+\rho_0)\hat{m}_0 \\
= & -\frac{1}{2}\hat{u}D_x^{-1}K(\hat{u},\hat{m}_0)-(\frac{1}{2}\rho_1+\rho_0)\pi_1 ad_A(D_x-\mathrm{ad}_{\hat{a}})\mathrm{ad}_A(\frac{1}{2}\rho_1+\rho_0)\hat{m}_0 \\
= & -\frac{1}{2}\hat{u}D_x^{-1}K(\hat{u},\hat{m}_0)-(\frac{1}{2}\rho_1+\rho_0)\pi_1 ad_A(D_x-\mathrm{ad}_{\hat{a}})\mathrm{ad}_A\pi_1(\frac{1}{2}\rho_1+\rho_0)\hat{m}_0 \\
= & \ \hat{\mathfrak{J}},
\end{aligned}
$$

in which we have used the fact that $(\frac{1}{2}\rho_1+\rho_0)\pi_1(\mathrm{ad}_{\hat{a}}^2\hat{u}) = -\hat{u}$ by applying the previous lemma. In the last line, we add π_1 at end to have symmetrized expression, since it would not change anything.

Thus replacing the last equation into the evolution (4.4.3), we get that

$$\hat{u}_t = \hat{\mathfrak{H}}\hat{\mathfrak{J}}\hat{m}_0+\hat{\mathfrak{A}}\hat{m}_0.$$

■

Remark 4.4.4. Similar results have been derived in the general case of Riemannian symmetric spaces in [2]. The Author has given a definition of parallel frame based on the choice of $\hat{a}$. It seems there is a gap if we compare two result. Indeed if we choose $\hat{a}$ as we have chosen here, then $\omega(D_x)$ will be determined according to his set up and that is not what we have. This needs further research.

Remark 4.4.5. This is exactly the Poisson operator in [69, 1.13] which in general is defined on Hermitian symmetric spaces. See also [70, 68]. The equation (4.4.2) corresponds to the λ coefficient of Lax representation, see Chapter 5.

Remark 4.4.6. For the related topics and similar construction, see [37, 75, 76, 61], and also [43]. For instance in [44], the authors apply a method of Sym and Pohlmeyer, [66, 54], to the Fordy-Kulish generalized nonlinear Schrodinger systems associated with Hermitian symmetric spaces [22]. Furthermore in [44], the authors gives also an appropriate specialization in the context of the symmetric space $SO(p+2)/SO(p)\times SO(2)$

which yields evolution equations for curves in $\mathbb{R}^{p+1}$ and S^p, with natural curvatures satisfying a generalized mKDV system. In fact this example is related to the constructions of Doliwa and Santini and illuminates certain features of the latter.

Remark 4.4.7. The rule of $\mathfrak{A}$ operator is very much similar to that of *interwining operator* $\mathcal{Z}$ defined as in [44, p. 166]. That is

$$\mathcal{Z}(Y) = \mathrm{ad}_A(\mathrm{Ad}_\phi Y)$$

for Y as Sym-Pohlmeyer field and is proved that

$$\mathcal{Z}\mathcal{R}Y = (\tilde{\mathcal{R}} - \lambda)\mathcal{Z}Y, \quad \mathcal{R}Y = -\mathcal{P}([T, \mathfrak{a}_s Y]), \tilde{\mathcal{R}}Y = (\mathfrak{a}_s - \mathrm{ad}_Q \mathfrak{a}_s^{-1} \mathrm{ad}_Q)\mathrm{ad}_A$$

in which $\mathcal{P}$ is defined as

$$\mathcal{P}(B) = \{\mathfrak{a}_s^{-1}[Q, B_m] + B_m\}.$$

There $\mathcal{R}$ and $\mathcal{P}$ are called respectively *geometric recursion operator* and *renormalization operator*. See the next section for the Lie algebraic form of recursion operator $\mathfrak{H}$ which is exactly of the form $\tilde{\mathcal{R}}$.