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Lax representation of an integrable system

5.1 Theory of Lax method

It is well known that most of the integrable nonlinear partial differential equations,

$$u_t = F(t, x, u, u_x, \dots, u_{nx}),$$

admit a Lax representation,

$$L_t = [A, L],$$

in which L, A are linear differential operators. Originally, the subject is due to the discovery by Gardner, Greene, Miura and Kruskal in [26] that the eigenvalues of the Schrödinger operator are integrals of the Korteweg-de Vries equation. At the same time, Lax presented a general principle for associating nonlinear evolution equations with linear operators so that the eigenvalues of the linear operator are integrals of the nonlinear equation, see [45]. To have a simple picture of the Lax construction, let \mathcal{B} be some Hilbert space of functions, for instance a space of smooth functions, chosen so that the function $u(t)$ lies in \mathcal{B} . We recall from functional analysis that a Hilbert space is a vector space with an inner product so that there can be defined a norm on the space and in addition it is complete. Suppose that to each function $u \in \mathcal{B}$, we can associate a self adjoint operator $L = L_u$ over some Hilbert space,

$$u \rightarrow L_u, \tag{5.1.1}$$

with the following property: If u changes with t subject to the equation

$$u_t = K(u),$$

the operators $L(t)$, which also change with t , remain unitary equivalent. If this is the case, then eigenvalues of L_u constitute a set of integrals for the equation under consideration. The unitary equivalence of the operators $L(t)$, mentioned above, means that there is a one-parameter family of unitary operators $U(t)$ such that

$$U(t)^{-1}L(t)U(t), \tag{5.1.2}$$

is independent of t . This fact can be expressed by setting the t derivative of (5.1.2) equal to zero:

$$-U^{-1}U_tU^{-1}LU + U^{-1}L_tU + U^{-1}LU_t = 0. \tag{5.1.3}$$

As a matter of fact, Stone's theorem [65], says that a one-parameter family of unitary operators on a Hilbert space satisfies a differential equation of the form

$$U_t = AU \quad (5.1.4)$$

where $A(t)$ is an antisymmetric operator. Conversely, every solution of (5.1.4) with $A^* = -A$ is a one-parameter family of unitary operators.

Now substituting (5.1.4) into (5.1.3) we obtain

$$L_t = AL - LA = [A, L]. \quad (5.1.5)$$

If u satisfies the equation $u_t = K(u)$, then L_t can be expressed in terms of u , and all that remains to verify is that equation (5.1.5) has an antisymmetric solution A . That means that the unitary equivalence of the operators $L(t)$ is nothing but finding the antisymmetric solution A for (5.1.5), which is called the Lax representation.

The drawback of this method is that it requires one to guess correctly the relation (5.1.1) between the function u and the operator L . What we will do later on is that we take the operator L depending to specific function u , then we will proceed by choosing proper operator, or ansatz, A , so that the Lax representation (5.1.5) hold from which we find the evolution equation.

To put some light on the subject, let us consider the Schrödinger operator

$$L = D^2 + \frac{1}{6}u.$$

Then L_t is multiplication by $\frac{1}{6}u_t$. Thus we have to find an antisymmetric operator A whose commutator with L is multiplication. To obtain a nontrivial result, let us choose $A_1 = D^3 + aD + Da$. The coefficient a is to be chosen. Now we have

$$[A_1, L] = \left(\frac{1}{2}u_x - 4a_x\right)D^2 + \left(\frac{1}{2}u_{xx} - 4a_{xx}\right)D + \frac{1}{6}u_{xxx} - a_{xxx} + \frac{1}{3}au_x.$$

Clearly, to eliminate all but the zero order terms we have to choose

$$a = \frac{1}{8}u.$$

With this choice, $[A_1, L]$ is multiplication by

$$\frac{1}{24}(u_{xxx} + uu_x).$$

Setting $A = 24A_1$, we verify that

$$[A, L] = K(u),$$

where $K(u) = u_t$ is KDV equation

$$u_t + u_{xxx} + uu_x = 0.$$

Later on, Drinfel'd and Sokolov in [16], showed that given the operator

$$L = D_x + a\lambda + q(x, t), \quad (5.1.6)$$

where λ is the so called **spectral parameter**, q belongs to a Lie algebras \mathfrak{g} , and a is a constant element of \mathfrak{g} , one can construct an operator $A = \sum_{i=0}^n p_i \lambda^i$ such that the Lax representation $L_t = [A, L]$ is equivalent to the evolution equation of the form

$$q_t = F(q, q_x, q_{xx}, \dots). \quad (5.1.7)$$

The method given in [16], which is based on bringing the operator L to diagonal form, allows one to constructively build, in addition to the A operator, the higher symmetries and integrals (conservation laws) for (5.1.7). The construction of [16], can be generalized to the case of L operators of a more general form:

$$L = D_x + a\lambda^{n+1} + \sum_{i=-m}^n q_i(x, t)\lambda^i.$$

In this chapter, we set up the method of Drinfel'd and Sokolov developed in [16] and [30]. The proofs here will be based on normal form theory for filtered Lie algebras. Then we will specialize to the case of the symplectic Lie algebra.

Definition 5.1.1. The Lie algebra $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ is called \mathbb{Z} -graded if \mathfrak{g}_i are vector space such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ where $i, j \in \mathbb{Z}$. It is clear that \mathfrak{g}_0 is a subalgebra of \mathfrak{g} .

Let us consider a \mathbb{Z} -graded Lie algebra $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ and an Lax operator of the form

$$L = D + \alpha + q(x, t), \quad q \in \mathfrak{g}_0. \quad (5.1.8)$$

Here $\alpha \in \mathfrak{g}_1$ satisfy the condition

$$\mathfrak{g} = \ker(ad_\alpha) \oplus \text{im}(ad_\alpha). \quad (5.1.9)$$

Indeed the condition 5.1.9 is an assumption on α .

Now let us define $\mathfrak{F}_n = \prod_{i=n}^{\infty} \tilde{\mathfrak{g}}_i$ where $\tilde{\mathfrak{g}}_i = \mathfrak{g}_{-i}$. Then \mathfrak{F}_n is a filtration. Let us denote α by α_{-1} which by definition must be in \mathfrak{F}_{-1} .

Lemma 5.1.1. Any element Z of the Lie algebra can be written as

$$Z = [\alpha_{-1}, X] + Y, \quad (5.1.10)$$

where $Y \in \ker(ad_{\alpha_{-1}})$ and $X \in \text{im}(ad_{\alpha_{-1}})$. Moreover if $Z \in \mathfrak{F}_n$ then $Y \in \mathfrak{F}_n$ and $X \in \mathfrak{F}_{n+1}$.

Proof. By decomposition there exist $Z^{(1)} \in \ker(ad_{\alpha_{-1}})$ and $Z^{(2)} \in \mathfrak{g}$ such that $Z = Z^{(1)} + [\alpha_{-1}, Z^{(2)}]$. Applying decomposition for $Z^{(2)}$, we find $Z^{(2)} = X^{(1)} + X^{(2)}$ with $X^{(1)} \in \ker(ad_{\alpha_{-1}})$ and $X^{(2)} \in \text{im}(ad_{\alpha_{-1}})$. Hence $[\alpha_{-1}, Z^{(2)}] = [\alpha_{-1}, X^{(2)}]$. Now we just take $X = X^{(2)}$.

Now suppose $Z \in \mathfrak{F}_n$. Then $[\alpha_{-1}, X], Y \in \mathfrak{F}_n$. Assume that $X \neq 0$ and k is lowest order that $X \in \mathfrak{F}_k$ and $X_k \in \tilde{\mathfrak{g}}_k$ is lowest nonzero term of it. Then $[\alpha_{-1}, X_k] \in \tilde{\mathfrak{g}}_{k-1} \cap \mathfrak{F}_n$. Now if $k-1 < n$, Then it follows that $[\alpha_{-1}, X_k]$ must be zero, i.e $X_k \in \ker(ad_{\alpha_{-1}})$. On the other hand $X \in \text{im}(ad_{\alpha_{-1}})$ and hence $X_k \in \text{im}(ad_{\alpha_{-1}})$. Thus $X_k \in \ker(ad_{\alpha_{-1}}) \cap \text{im}(ad_{\alpha_{-1}}) = 0$ which is in contradiction with the assumption that X_k is lowest nonzero term of X . Hence we conclude that $k-1 \geq n$, and so $X \in \mathfrak{F}_k \subseteq \mathfrak{F}_{n+1}$. \blacksquare

Lemma 5.1.2. Let us define the following maps

$$\mathfrak{F}_k/\mathfrak{F}_{k+1} \xrightarrow{\alpha_{-1}^k} \mathfrak{F}_{k-1}/\mathfrak{F}_k \xrightarrow{\alpha_{-1}^{k-1}} \mathfrak{F}_{k-2}/\mathfrak{F}_{k-1}.$$

Now we have that

$$\mathfrak{F}_{k-1}/\mathfrak{F}_k = \ker(\alpha_{-1}^{k-1}) \oplus \text{im}(\alpha_{-1}^k).$$

Proof. We see that

$$\ker(\alpha_{-1}^{k-1}) = \{Z + \mathfrak{F}_k \mid Z \in \mathfrak{F}_{k-1}, [\alpha_{-1}, Z] \in \mathfrak{F}_{k-1}\},$$

and if Z^{k-1} is first term of Z , then that means that $[Z^{k-1}, \alpha_{-1}] = 0$. Also we have that

$$\text{im}(\alpha_{-1}^k) = \{Z + \mathfrak{F}_k \mid \exists X \in \mathfrak{F}_k \text{ such that } Z - [\alpha_{-1}, X] \in \mathfrak{F}_k\}.$$

This means that $Z^{k-1} = [\alpha_{-1}, X^k]$. Now suppose $Z \in \mathfrak{F}_n$, Then by the general decomposition there exist $Y \in \ker(ad_{\alpha_{-1}})$ and $X \in \text{im}(ad_{\alpha_{-1}})$ such that $Z = Y + [\alpha_{-1}, X]$. It is clear that $Y, [\alpha_{-1}, X] \in \mathfrak{F}_n$. Hence

$$Z + \mathfrak{F}_{n+1} = Y + \mathfrak{F}_{n+1} + [\alpha_{-1}, X] + \mathfrak{F}_{n+1}.$$

Obviously $[\alpha_{-1}, Y] = 0 \in \mathfrak{F}_n$, hence $Y + \mathfrak{F}_{n+1} \in \ker(\alpha_{-1}^{n-1})$. Now suppose $X \in \mathfrak{F}_k$. If $k \geq n+1$ then $X \in \mathfrak{F}_{n+1}$. Hence

$$[\alpha_{-1}, X] - [\alpha_{-1}, X] = 0 \in \mathfrak{F}_{n+1}.$$

Thus $[\alpha_{-1}, X] + \mathfrak{F}_{n+1} \in \text{im}(\alpha_{-1}^n)$. Therefore

$$Z + \mathfrak{F}_{n+1} = Y + \mathfrak{F}_{n+1} + [\alpha_{-1}, X] + \mathfrak{F}_{n+1},$$

in which $Y + \mathfrak{F}_{n+1} \in \ker(\alpha_{-1}^{n-1})$ and $[\alpha_{-1}, X] + \mathfrak{F}_{n+1} \in \text{im}(\alpha_{-1}^n)$. Now suppose $k < n+1$ and $X = \hat{X} + \tilde{X}$ in which $\hat{X} \in \mathfrak{F}_{n+1}$. But then

$$[\alpha_{-1}, X] + \mathfrak{F}_{n+1} = [\alpha_{-1}, \hat{X}] + [\alpha_{-1}, \tilde{X}] + \mathfrak{F}_{n+1}.$$

Now since $[\alpha_{-1}, X], [\alpha_{-1}, \tilde{X}] \in \mathfrak{F}_n$ and that $[\alpha_{-1}, \hat{X}] \in \mathfrak{F}_{k-1} - \mathfrak{F}_{n-1}$ so

$$[\alpha_{-1}, \hat{X}] = 0.$$

Thus

$$[\alpha_{-1}, X] + \mathfrak{F}_{n+1} = [\alpha_{-1}, \tilde{X}] + \mathfrak{F}_{n+1}.$$

Again $[\alpha_{-1}, \tilde{X}] - [\alpha_{-1}, \tilde{X}] = 0 \in \mathfrak{F}_{n+1}$ in which $\tilde{X} \in \mathfrak{F}_{n+1}$. Therefore $[\alpha_{-1}, \tilde{X}] + \mathfrak{F}_{n+1} \in \text{im}(\alpha_{-1}^n)$. Hence

$$Z + \mathfrak{F}_{n+1} = Y + \mathfrak{F}_{n+1} + [\alpha_{-1}, \tilde{X}] + \mathfrak{F}_{n+1},$$

in which $Y + \mathfrak{F}_{n+1} \in \ker(\alpha_{-1}^{n-1})$ and $[\alpha_{-1}, \tilde{X}] + \mathfrak{F}_{n+1} \in \text{im}(\alpha_{-1}^n)$. \blacksquare

Now we give a second proof of Lemma 5.1.1 using the previous lemma: Let us Suppose that $Z_{k-1} \in \mathfrak{F}_{k-1}$. Then by the previous lemma we have that

$$Z_{k-1} + \mathfrak{F}_k = Y_{k-1} + \mathfrak{F}_k + W_{k-1} + \mathfrak{F}_k,$$

where $Y_{k-1} + \mathfrak{F}_k \in \ker(\alpha_{-1}^{k-1})$ and $W_{k-1} + \mathfrak{F}_k \in \text{im}(\alpha_{-1}^k)$. Therefore $W - [\alpha_{-1}, X_k] \in \mathfrak{F}_k$ in which $X_k \in \mathfrak{F}_k$ and $[\alpha_{-1}, Y_{k-1}] \in \mathfrak{F}_{k-1}$. Here we can suppose that $X_k \in \mathfrak{F}_k \cap \text{im}(ad_{\alpha_{-1}})$. On the other hand, we have that $[\alpha_{-1}, Y_{k-1}^{k-1}] = 0$. Hence

$$Z_{k-1} + \mathfrak{F}_k = Y_{k-1} + [\alpha_{-1}, X_k] + \mathfrak{F}_k = Y_{k-1}^{k-1} + [\alpha_{-1}, X_k] + \mathfrak{F}_k.$$

Thus

$$Z_{k-1} = Y_{k-1}^{k-1} + [\alpha_{-1}, X_k] + \hat{Z}_k.$$

Similarly we have

$$\hat{Z}_k = Y_k^k + [\alpha_{-1}, X_{k+1}] + \hat{Z}_{k+1},$$

where again $[\alpha_{-1}, Y_k^k] = 0$ and $X_{k+1} \in \mathfrak{F}_k \cap \text{im}(ad_{\alpha_{-1}})$. Hence we get that

$$Z_{k-1} = Y_{k-1}^{k-1} + Y_k^k + \dots + [\alpha_{-1}, X_k + X_{k+1} + \dots] = Y + [\alpha_{-1}, X].$$

So is clear that $Y \in \mathfrak{F}_{k-1} \cap \ker(ad_{\alpha_{-1}})$ and $X \in \mathfrak{F}_k \cap \text{im}(ad_{\alpha_{-1}})$.

Remark 5.1.3. In the notation of [56], $L_0 = D + \alpha_{-1} + q_{(0)}$ is in \mathfrak{F}_1^0 and $u_1 \in \mathfrak{F}_0^1$. In general $L = D + \alpha_{-1} + q \in \mathfrak{F}_1^0$ and $u \in \mathfrak{F}_0^1$ and $L_0 = D + \alpha_{-1} + h \in \mathfrak{F}_1^0$ and L_0 and L are in the same equivalence class and L_0 is normal form for that class. For more information about normal form theory, see [56].

The following proposition ([30]) plays a key rule in constructing an integrable equation using a Lax pair.

Proposition 5.1.4. There exist an element $u = \sum_1^\infty u_i, u_i \in \text{im}(ad_{\alpha_{-1}}) \cap \tilde{\mathfrak{g}}_i$, and $h = \sum_0^\infty h_i, h_i \in \ker(ad_{\alpha_{-1}}) \cap \tilde{\mathfrak{g}}_i$ such that

$$e^{adu} L = D + \alpha_{-1} + h.$$

Proof. The claim is that for each $n \in \mathbb{N} \cup \{0, -1\}$ there exist

$$u_n \in \mathfrak{F}_n \cap \text{im}(ad_{\alpha_{-1}}), \quad \text{and} \quad q_{(n)} \in \ker(ad_{\alpha_{-1}}) \cap \mathfrak{F}_0 + \mathfrak{F}_n,$$

such that

$$e^{ad_{u_n}}(L_n) = L_{n+1}, \quad \text{where} \quad L_n = D + \alpha_{-1} + q_{(n)}.$$

When $n = -1$ then we take $u_{-1} = 0 \in \mathfrak{F}_0 \cap \text{im}(ad_{\alpha_{-1}})$ then $L_{-1} = L_0 = D + \alpha_{-1} + q_{(-1)}$ with $q_{(-1)} = q_{(0)} = q \in \mathfrak{F}_0 \subseteq \mathfrak{F}_0 + \ker(ad_{\alpha_{-1}}) \cap \mathfrak{F}_0$. Now suppose the statement holds for n . Then by assumption we can write

$$q_{(n)} = \bar{q}_{(n)} + y_n,$$

where $\bar{q}_{(n)} \in \ker(ad_{\alpha_{-1}}) \cap \mathfrak{F}_0$ and $y_n \in \mathfrak{F}_n$. Now by semisimple decomposition

$$y_n = \bar{y}_n + [\alpha_{-1}, u_{n+1}],$$

where $\bar{y}_n \in \ker ad_{\alpha_{-1}} \cap \mathfrak{F}_n \subset \ker ad_{\alpha_{-1}} \cap \mathfrak{F}_0$ and $u_{n+1} \in \mathfrak{F}_{n+1} \cap \text{im}(ad_{\alpha_{-1}})$ by Lemma 5.1.1. It follows that

$$\begin{aligned} D + \alpha_{-1} + q_{(n+1)} &= e^{ad_{u_{n+1}}}(D + \alpha_{-1} + q_{(n)}) \\ &= D + \alpha_{-1} + q_{(n)} + [u_{n+1}, \alpha_{-1}] \pmod{\mathfrak{F}_{n+1}} \\ &= D + \alpha_{-1} + \bar{q}_{(n)} + \bar{y}_n \pmod{\mathfrak{F}_{n+1}}. \end{aligned}$$

Hence

$$q_{(n+1)} = \bar{q}_{(n)} + \bar{y}_n \pmod{\mathfrak{F}_{n+1}}.$$

This indeed means that $q_{(n+1)} \in \ker ad_{\alpha_{-1}} \cap \mathfrak{F}_0 + \mathfrak{F}_{n+1}$. Thus the proof of the induction statement is complete. By the Campbell-Baker-Hausdorff formula there exist

$$\tilde{u}_{(n)} \in \mathfrak{F}_1 \cap \text{im}(ad_{\alpha_{-1}}),$$

such that

$$e^{ad_{\tilde{u}_{(n)}}} L_0 = e^{ad_{u_n}} e^{ad_{u_{n-1}}} \dots e^{ad_{u_0}} L_{-1} = L_n = D + \alpha_{-1} + q_{(n)}.$$

Since in our case $\bigcap \mathfrak{F}_n = 0$, it follows that in the limit $n \rightarrow \infty$ one has

$$e^{ad_{\tilde{u}_\infty}}(D + \alpha_{-1} + q) = D + \alpha_{-1} + q_\infty. \quad \blacksquare$$

Notice that according to [30], we are supposed to choose $\mathfrak{A}_+, \mathfrak{A}_-$ in such a way that

$$\mathfrak{g} = \mathfrak{A}_+ \oplus \mathfrak{A}_-, \quad \text{and} \quad \mathfrak{A}_- \subseteq \ker(ad_{A_\lambda}).$$

Now lets set

$$\bar{\mathfrak{g}}_+ = \bigoplus_{i>0} \mathfrak{g}_i + \mathfrak{A}_+, \quad \bar{\mathfrak{g}}_- = \bigoplus_{i<0} \mathfrak{g}_i + \mathfrak{A}_-. \quad (5.1.11)$$

We use grading and not filtering from now on. The next result shows that L_t and $[A_\beta^+, L]$ sit in the same space. We show this in more general form of Lax operator as below. One can see that the previous theorem holds in this case as well.

Proposition 5.1.5. Consider the Lax operator $L = D + \mathbf{A}\lambda^{n+1} + \sum_{i=-m}^n q_i \lambda^i$ and let β be a constant from the center of subalgebra $\ker(ad_{-(\alpha_{n+1})})$,

$$A_\beta = e^{-ad_u} \beta, \quad \text{and} \quad A_\beta^+ = (e^{-ad_u} \beta)_+,$$

where “+” denotes the projection onto $\bar{\mathfrak{g}}_+$ parallel to $\bar{\mathfrak{g}}_-$. Then $[A_\beta^+, L] \in \sum_{i=-m}^n \mathfrak{g}_i$.

Proof. We have

$$\begin{aligned} [e^{-ad_u} \beta, L] &= [e^{-ad_u} \beta, e^{-ad_u} (D + \alpha_{n+1} + h)] = e^{-ad_u} [\beta, D + \alpha_{n+1} + h] \\ &= e^{-ad_u} (-\beta_x + [\beta, \alpha_{n+1}] + [\beta, h]) = e^{-ad_u} (0) = 0, \end{aligned}$$

using the assumption that β is in the center of $\text{Ker}(ad_{\alpha_{n+1}})$ and that $\alpha_{-(n+1)}, h_i \in \text{Ker}(ad_{\alpha_{n+1}})$. Hence

$$[(e^{-ad_u} \beta)_+, L] = -[(e^{-ad_u} \beta)_-, L].$$

Now $[(e^{-ad_u} \beta)_+, L]$ sits in $\sum_{i=-m}^\infty \mathfrak{g}_i$ and $[(e^{-ad_u} \beta)_-, L]$ in $\sum_{i=-\infty}^n \mathfrak{g}_i$, hence $[(e^{-ad_u} \beta)_+, L]$ sits in intersection space which is $\sum_{i=-m}^n \mathfrak{g}_i$. Indeed if $e^{-ad_u} \beta = \sum_{i=-\infty}^1 X_i + X_- + \sum_{i=1}^\infty X_i + X_+$ where $X_i \in \mathfrak{g}_i, X_- \in \mathfrak{A}_-$ and $X_+ \in \mathfrak{A}_+$, then we have that

$$\begin{aligned} [(e^{-ad_u} \beta)_+, L] &= \left[\sum_1^\infty X_i + X_+, L \right] \\ &= \sum_1^\infty (-X_{i,x} + [X_i, \alpha_{n+1}] + \sum_{j=-m}^n [X_i, q_j]) - X_{+,x} + [X_+, \alpha_{n+1}] \\ &\quad + \sum_{j=-m}^n [X_+, q_j] \\ &\in \sum_{-m}^\infty \mathfrak{g}_i, \end{aligned}$$

and

$$\begin{aligned} [(e^{-ad_u} \beta)_-, L] &= \left[\sum_{-\infty}^{-1} X_i + X_-, L \right] \\ &= \sum_{-\infty}^{-1} (-X_{i,x} + [X_i, \alpha_{n+1}] + \sum_{j=-m}^n [X_i, q_j]) - X_{-,x} + [X_-, \alpha_{n+1}] + \sum_{j=-m}^n [X_-, q_j] \\ &= \sum_{-\infty}^{-1} (-X_{i,x} + [X_i, \alpha_{n+1}] + \sum_{j=-m}^n [X_i, q_j]) - X_{-,x} \\ &\quad + \sum_{j=-m}^n [X_-, q_j] \\ &\in \sum_{-\infty}^n \mathfrak{g}_i, \end{aligned}$$

using the fact that $\mathfrak{A}_- \subset \ker(ad_{\alpha_{n+1}})$. Hence $[(e^{-ad_u}\beta)_+, L] \in \sum_{-m}^n \mathfrak{g}_i$. ■

Proof. second proof: Here $L = D + \alpha_{-(n+1)} + q$ where $q \in \mathfrak{F}_{-n}/\mathfrak{F}_{(m+1)}$. Let us define a few notations. We denote $\mathfrak{F}_k/\mathfrak{F}_{k+1}$ by \mathfrak{G}_k and $\mathfrak{G}_k + \dots + \mathfrak{G}_l$ by \mathfrak{H}_k^l where $k < l$. We want to prove that

$$[(e^{-ad_u}\beta)_+, L] \in \mathfrak{H}_{-n}^m.$$

We see that

$$\begin{aligned} [e^{-ad_u}\beta, L] &= [e^{-ad_u}\beta, e^{-ad_u}(D + \alpha_{n+1} + h)] \\ &= e^{-ad_u}[\beta, D + \alpha_{n+1} + h] \\ &= e^{-ad_u}(-\beta_x + [\beta, \alpha_{n+1}] + [\beta, h]) \\ &= e^{-ad_u}(0) = 0, \end{aligned}$$

using the assumption that β is in the center of $\ker(ad_{\alpha_{-(n+1)}})$ and that $\alpha_{-(n+1)}, h_i \in \ker(ad_{\alpha_{n+1}})$. Hence

$$[(e^{-ad_u}\beta)_+, L] = -[(e^{-ad_u}\beta)_-, L].$$

Let us put $(e^{-ad_u}\beta)_- = X + X_-$ in which $X_- \in \mathfrak{A}_-$ and $X \in \mathfrak{F}_1 = \mathfrak{H}_1^\infty$. Also $(e^{-ad_u}\beta)_+ = Y + X_+$ in which $X_+ \in \mathfrak{A}_+$ and $Y \in \mathfrak{H}_{-\infty}^{-1}$. Then we have that

$$\begin{aligned} [(e^{-ad_u}\beta)_-, L] &= -X_x - X_{-,x} + [X, \alpha_{-(n+1)}] + [X_-, \alpha_{-(n+1)}] \\ &\quad + [X, q] + [X_-, q] \\ &= -X_x - X_{-,x} + [X, \alpha_{-(n+1)}] + [X, q] + [X_-, q]. \end{aligned}$$

Here $[X, \alpha_{-(n+1)}] \in \mathfrak{H}_{-n}^\infty$, and $[X, q] \in \mathfrak{H}_{-n+1}^\infty$. Hence $[(e^{-ad_u}\beta)_-, L] \in \mathfrak{H}_{-n}^\infty$. Similarly

$$\begin{aligned} [(e^{-ad_u}\beta)_+, L] &= -Y_x - X_{+,x} + [Y, \alpha_{-(n+1)}] + [X_+, \alpha_{-(n+1)}] \\ &\quad + [Y, q] + [X_+, q]. \end{aligned}$$

Here $[Y, \alpha_{-(n+1)}] \in \mathfrak{H}_{-\infty}^{-1}$ and $[Y, q] \in \mathfrak{H}_{-\infty}^{m-1}$ and also $[X_+, q] \in \mathfrak{H}_{-n}^m$. Hence we have that $[(e^{-ad_u}\beta)_+, L] \in \mathfrak{H}_{-\infty}^m$. Therefore $[(e^{-ad_u}\beta)_+, L] \in \mathfrak{H}_{-\infty}^m \cap \mathfrak{H}_{-n}^\infty = \mathfrak{H}_{-n}^m$. ■

Remark 5.1.6. Instead of \mathfrak{H}_p^q we may use the notation in [56] as $f^{p|q}$ and call it q -jet of f^p .

The claim of proposition above is that relation $L_t = [A_\beta^+, L]$ is equivalent to some evolution system for the unknown q . In the case we will work out, the operator L is taken to be $L = D + \alpha + q$ with $q \in \mathfrak{G}_0$ and $\alpha_{-1} = \mathbf{A}\lambda \in \mathfrak{G}_{-1}$.

Proposition 5.1.7. Let $M \in \mathfrak{F}_{-n}$ in which $n > 0$. Suppose that we have

$$[D_t - M, L_0] = 0 \quad \text{or} \quad L_{0,t} = [M, L_0], \tag{5.1.12}$$

where $L_0 = D_x + \alpha_{-1} + h$ and $h \in \ker(ad_{\alpha_{-1}}) \cap \mathfrak{F}_0$. Then $M \in \ker(ad_{\alpha_{-1}})$.

Proof. We see that

$$h_t = [M^{-n}, \alpha_{-1}] \pmod{\mathfrak{F}_{-n}}.$$

Thus $[M^{-n}, \alpha_{-1}] \in \text{Ker}(ad_{\alpha_{-1}}) + \mathfrak{F}_{-n}$. Hence $[M^{-n}, \alpha_{-1}] \in \ker(ad_{\alpha_{-1}})$ and so $M^{-n} \in \ker(ad_{\alpha_{-1}})$. Thus $M \in \ker(ad_{\alpha_{-1}}) + \mathfrak{F}_{-n+1}$. Now suppose $M^{-n+i} \in \text{Ker}(ad_{\alpha_{-1}})$ for $i = 0, \dots, k-1$. Hence

$$M = \hat{M} + \tilde{M},$$

in which $\hat{M} \in \text{Ker}(ad_{\alpha_{-1}})$ and $\tilde{M} \in \mathfrak{F}_{-n+k}$. From (5.1.12), we have that

$$h_t + \hat{M}_x - [\hat{M}, h] = -\tilde{M}_x + [\tilde{M}, \alpha_{-1}] + [\tilde{M}, h].$$

Since $h_t + \hat{M}_x - [\hat{M}, h] \in \ker(ad_{\alpha_{-1}})$. The same reasoning shows that $M^{-n+k} = \tilde{M}^{-n+k} \in \ker(ad_{\alpha_{-1}})$. Hence by the filtering topology in \mathfrak{F}'_n s we get that $M \in \ker(ad_{\alpha_{-1}})$. \blacksquare

Lemma 5.1.8. 1. Let β and γ be arbitrary elements of the center of subalgebra $\ker(ad_{A\lambda})$. Consider the equation

$$L_t = [A_\beta^+, L], \quad (5.1.13)$$

$$\text{then } \frac{dA_\gamma}{dt} = [A_\beta^+, A_\gamma].$$

2. $[A_\beta^+, A_\gamma]_+ = [A_\gamma, (A_\beta)_-]_+ = [A_\gamma^+, (A_\beta)_-]_+$. Notice that these identities does not depend on (5.1.13).

Proof. From (5.1.13) we obtain

$$0 = [d_t - A_\beta^+, L] = [d_t - A_\beta^+, e^{-ad_u} L_0] = e^{-ad_u} [e^{ad_u} (d_t - A_\beta^+), L_0],$$

thus

$$0 = [e^{ad_u} (d_t - A_\beta^+), L_0] = [d_t - \hat{A}_\beta^+, L_0],$$

where $L_0 = D_x + \alpha + \sum_{-\infty}^0 h_i$ in which $h_i \in \ker(ad_\alpha) \cap \mathfrak{g}_i$ and $\alpha \in \mathfrak{g}_1$. From the last proposition we know that $\hat{A}_\beta^+ \in \ker(ad_\alpha)$. Now we want to prove that $[d_t - A_\beta^+, A_\gamma] = 0$ or

$$[d_t - \hat{A}_\beta^+, \gamma] = 0,$$

or

$$[\hat{A}_\beta^+, \gamma] = 0,$$

since γ is a constant. Now since γ is in the center of $\ker(ad_\alpha)$ and \hat{A}_β^+ in the $\ker(ad_\alpha)$, therefore this equality is trivial. To prove (2), one notice that

$$\begin{aligned} [A_\beta^+, A_\gamma] &= [(e^{-ad_u} \beta)_+, e^{-ad_u} \gamma] \\ &= [e^{-ad_u} \beta - (e^{-ad_u} \beta)_-, e^{-ad_u} \gamma] \\ &= [e^{-ad_u} \beta, e^{-ad_u} \gamma] + [e^{-ad_u} \gamma, (e^{-ad_u} \beta)_-] \\ &= e^{-ad_u} [\beta, \gamma] + [e^{-ad_u} \gamma, (e^{-ad_u} \beta)_-] \\ &= [e^{-ad_u} \gamma, (e^{-ad_u} \beta)_-] \\ &= [A_\gamma, (A_\beta)_-], \end{aligned}$$

since β, γ are in the center of $\ker(ad_\alpha)$ as well as in $\ker(ad_\alpha)$ itself clearly. \blacksquare

Theorem 5.1.9. Let β and γ be two arbitrary elements of the center of subalgebra $\ker(ad_{A\lambda})$. Then the flows

$$L_t = [A_\beta^+, L], \quad \text{and} \quad L_\tau = [A_\gamma^+, L],$$

commute with each other, i.e.,

$$D_{[A_\beta^+, L]}[A_\gamma^+, L] - D_{[A_\gamma^+, L]}[A_\beta^+, L] = 0, \quad \text{or} \quad \text{equivalently,} \quad (L_t)_\tau = (L_\tau)_t.$$

In particular the flows $L_{t_n} = [A_{\beta\lambda^n}^+, L]$ will commute. One can choose $\beta = \alpha$, since α and $\alpha\lambda^n = A\lambda^{n+1}$ are naturally in the center of $\ker(ad_{A\lambda})$.

Proof. We have

$$\frac{d}{d\tau} \left(\frac{dL}{dt} \right) = \frac{d}{d\tau} [A_\beta^+, L] = \left[\frac{d}{d\tau} A_\beta^+, L \right] + [A_\beta^+, \frac{d}{d\tau} L].$$

According to Lemma 5.1.8,

$$\frac{d}{d\tau} A_\beta^+ = [A_\gamma^+, A_\beta^+].$$

So $\frac{d}{d\tau} A_\beta^+ = [A_\gamma^+, A_\beta^+]_+$. Thus

$$\frac{d}{d\tau} \left(\frac{dL}{dt} \right) = [[A_\gamma^+, A_\beta^+]_+, L] + [A_\beta^+, [A_\gamma^+, L]].$$

Similarly $\frac{d}{dt} A_\gamma^+ = [A_\beta^+, A_\gamma^+]_+$ and that

$$\frac{d}{dt} \left(\frac{dL}{d\tau} \right) = [[A_\beta^+, A_\gamma^+]_+, L] + [A_\gamma^+, [A_\beta^+, L]].$$

Using Jacobi identity, we obtain

$$\begin{aligned} \frac{d}{d\tau} \left(\frac{dL}{dt} \right) - \frac{d}{dt} \left(\frac{dL}{d\tau} \right) &= \left[\frac{d}{d\tau} A_\beta^+ - \frac{d}{dt} A_\gamma^+ + [A_\beta^+, A_\gamma^+], L \right] \\ &= [[A_\gamma^+, A_\beta^+]_+ - [A_\beta^+, A_\gamma^+]_+ + [A_\beta^+, A_\gamma^+], L] \end{aligned}$$

But

$$[A_\gamma^+, A_\beta^+]_+ = [A_\beta, (A_\gamma)_-]_+ = [A_\beta^+, (A_\gamma)_-]_+$$

as in the previous lemma. Therefore

$$\begin{aligned} & [A_\gamma^+, A_\beta^+]_+ - [A_\beta^+, A_\gamma^+]_+ + [A_\beta^+, A_\gamma^+] \\ &= [A_\beta^+, (A_\gamma)_-]_+ - [A_\beta^+, A_\gamma^+]_+ + [A_\beta^+, A_\gamma^+]_+ \\ &= [A_\beta^+, -A_\gamma^+]_+ + [A_\beta^+, A_\gamma^+]_+ \\ &= [A_\beta^+, -A_\gamma^+ + A_\gamma^+]_+ = 0. \end{aligned}$$

Hence

$$\frac{d}{d\tau} \left(\frac{dL}{dt} \right) - \frac{d}{dt} \left(\frac{dL}{d\tau} \right) = 0.$$

■

Now we give some well known examples of \mathbb{Z} -graded Lie algebras.

Example 5.1.1. The first example is the Lie algebra of the **Laurent series**

$$\mathfrak{g} = \mathfrak{b}[[\lambda, \lambda^{-1}]] = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i,$$

where \mathfrak{g}_i consist of elements $A\lambda^i$ in which A belongs to the Lie algebra \mathfrak{b} . Here \mathfrak{g} is decomposed into sum of a polynomial in λ and a series containing only negative powers of λ . The instance of this decomposition in the set up above is (5.1.11). Notice that there we have taken $\mathfrak{A}_- = 0$ and $\mathfrak{A}_+ = \mathfrak{b}$. See [16] to get more on this specific \mathbb{Z} -graded Lie algebras.

Example 5.1.2. The second example is known as Kac-Moody algebras which indeed assign a \mathbb{Z} -graded Lie algebra to any automorphism of finite order on a given Lie algebra and is constructed as follows.

Let σ be any automorphism of finite order m on a finite dimensional Lie algebra \mathfrak{b} . By \mathfrak{b}_i we denote a subspace of the space \mathfrak{b} , consist of elements like $X \in \mathfrak{b}$ so that $\sigma(X) = \mu^i X$, where μ is a primitive m -root of unity. We have

$$\mathfrak{b} = \bigoplus_{i \in \mathbb{Z}_m} \mathfrak{b}_i, \quad (5.1.14)$$

so that $[\mathfrak{b}_i, \mathfrak{b}_j] \subset \mathfrak{b}_{i+j}$. We say that \mathfrak{b} is graduated modulo m . Now, to a Lie algebra (5.1.14), we assign a graduated Lie algebra $L(\mathfrak{b}, \sigma)$, by considering Laurent series

$$L(\mathfrak{b}, \sigma) = \mathfrak{b}[[\lambda, \lambda^{-1}]] = \bigoplus_{i \in \mathbb{Z}} \mathfrak{b}\lambda^i,$$

and taking in it a subalgebra

$$L(\mathfrak{b}, \sigma) = \bigoplus_{i \in \mathbb{Z}} \lambda^i \mathfrak{b}_i,$$

where \mathfrak{b}_i will be taken as modulo m . We denote \mathfrak{g}_i to $\mathfrak{b}_i\lambda^i$, so that $L(\mathfrak{b}, \sigma) = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$. For more information, see the work of V. G. Kac in [34].

Proposition 5.1.10. Suppose that the automorphism σ on the Lie algebra \mathfrak{b} is irreducible, i.e., automorphism with respect to which the algebra can not be decomposed into a direct sum of invariant ideals. Then there exist elements

$$E_0, \dots, E_r \in \mathfrak{g}_1, \quad F_0, \dots, F_r \in \mathfrak{g}_{-1}, \quad \text{and} \quad H_0, \dots, H_r \in \mathfrak{g}_0,$$

such that

1. The sets $\{E_0, \dots, E_r\}$ and $\{F_0, \dots, F_r\}$ form a basis in \mathfrak{g}_1 and \mathfrak{g}_{-1} respectively and the set $\{H_0, \dots, H_r\}$ generates \mathfrak{g}_0 .
2. The following relations hold:
 - (a) $[H_i, H_j] = 0$,
 - (b) $[E_i, F_j] = \delta_{ij}H_i$,
 - (c) $[H_i, E_i] = A_{ii}E_j$,
 - (d) $[H_i, F_j] = -A_{ij}F_j$,
 where $A_{ii} = 2$ for all i .

We use Theorem 5.1.9 to construct a recursion operator for a certain integrable evolution equations. Let us specifically work on Kac-Moody algebras \mathfrak{g} defined on the Lie algebra \mathfrak{b} . Assume that the the degree of automorphism σ is k . Let us take $\alpha = A\lambda$ in which A is constant element of \mathfrak{b} so that the decomposition (5.1.9) holds. Now we choose a sequence of constant elements of center of $\ker(ad_\alpha)$ as $\{A\lambda^n\}_{n \in \mathbb{N}}$. Then we would get a sequence of commuting equation or hierarchy of integrable equation. Here

$$A_{A\lambda^{n+k}}^+ = (e^{-ad_u} A\lambda^{n+k})_+ = \lambda^k (e^{-ad_u} A\lambda^n)_+ + (\lambda^k (e^{-ad_u} A\lambda^n)_-)_+.$$

Hence

$$L_{t_{n+k}} = [A_{A\lambda^{n+k}}^+, L] = \lambda^k L_{t_n} + [R_k, L], \quad (5.1.15)$$

where $R_k = (\lambda^k (e^{-ad_u} A\lambda^n)_-)_+$ is clearly of degree k and must be taken as the polynomial

$$R_k = N_k \lambda^k + N_{k-1} \lambda^{k-1} + \dots + N_0, \quad N_i \in \mathfrak{g}^i. \quad (5.1.16)$$

as an element of Kac-Moody algebras \mathfrak{g} .

Equating coefficients of λ powers of both side of (5.1.15) we will get hierarchy of equations as well as the Recursion operator as we can see in the section below for Symplectic Lie algebras of quaternions.

5.2 Lax method in Symplectic geometry

Let us consider the Lie algebra of the Symplectic group $\mathfrak{b} = \mathfrak{sp}(n+1)$. We first build up a Kac-Moody algebras \mathfrak{g} defined on the Lie algebra \mathfrak{b} . To be specific we define the automorphism σ as follows:

$$\sigma(X) = T' X T'^{-1},$$

where

$$T' = \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix}.$$

Obviously $\sigma^2 = Id$ and eigenvalues of σ are 1 and -1 , Indeed in this particular case, $\mu = -1$. Hence

$$\mathfrak{g}_{2i} = \mathfrak{g}^e \lambda^{2i}, \quad \mathfrak{g}_{2i+1} = \mathfrak{g}^o \lambda^{2i+1},$$

where

$$\mathfrak{g}^e = \begin{pmatrix} * & 0 & \mathbf{0} \\ 0 & * & * \\ \mathbf{0} & * & * \end{pmatrix}, \quad \mathfrak{g}^o = \begin{pmatrix} 0 & * & * \\ * & 0 & \mathbf{0} \\ * & \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

We immediately see that if we let $\mathfrak{a} = \mathfrak{sp}(n) \times \mathfrak{sp}(1)$, then

$$\mathfrak{g}^e = \mathfrak{a}, \quad \text{and} \quad \mathfrak{g}^o = \mathfrak{b}/\mathfrak{a}.$$

Let us choose

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & \mathbf{0} \\ -1 & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \in \mathfrak{g}^o.$$

In the next proposition we show that the Kac-Moody algebra decompose into the kernel and image of adjoint representation as in (5.1.9).

Proposition 5.2.1. Following decomposition holds.

$$\mathfrak{g} = \ker(ad_{A\lambda}) \oplus \text{im}(ad_{A\lambda}).$$

Moreover

$$\ker(ad_{A\lambda}) = \bigoplus_i \begin{pmatrix} 0 & t & \mathbf{0} \\ -t & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \lambda^{2i+1} \bigoplus_i \bigoplus_i \begin{pmatrix} p & 0 & \mathbf{0} \\ 0 & p & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{P} \end{pmatrix} \lambda^{2i},$$

where t is real number, p a pure quaternionic, and $\mathbf{P} \in \mathfrak{sp}(n-1)$. Also

$$\text{im}(ad_{A\lambda}) = \bigoplus_i \begin{pmatrix} 0 & p & -\bar{\mathbf{p}}^T \\ p & 0 & \mathbf{0} \\ \mathbf{p} & \mathbf{0} & \mathbf{0} \end{pmatrix} \lambda^{2i+1} \bigoplus_i \bigoplus_i \begin{pmatrix} q & 0 & \mathbf{0} \\ 0 & -q & -\bar{\mathbf{q}}^T \\ \mathbf{0} & \mathbf{q} & \mathbf{0} \end{pmatrix} \lambda^{2i}.$$

where p, q are pure quaternionic numbers.

Proof. Let us take general element \mathbf{M} of the basic Lie algebra \mathfrak{b} . Thus M has following form.

$$M = \begin{pmatrix} m_{11} & m_{12} & -\bar{\mathbf{m}}_1^T \\ m_{21} & m_{22} & -\bar{\mathbf{m}}_2^T \\ \mathbf{m}_1 & \mathbf{m}_2 & \mathbf{M} \end{pmatrix}, \quad (5.2.1)$$

in which $m_{21} = -\bar{m}_{12}^T$ and m_{11}, m_{22} are pure quaternionic numbers. Hence the bracket of \mathbf{M} and \mathbf{A} is

$$[\mathbf{M}, \mathbf{A}] = \begin{pmatrix} -m_{12} - m_{21} & m_{11} - m_{22} & \bar{\mathbf{m}}_2^T \\ m_{11} - m_{22} & m_{21} + m_{12} & -\bar{\mathbf{m}}_1^T \\ -\mathbf{m}_2 & \mathbf{m}_1 & 0 \end{pmatrix}. \quad (5.2.2)$$

Since $m_{11} - m_{22}$ and $m_{12} + m_{21}$ are pure quaternionic number, $\text{im}(ad_{A\alpha})$ is exactly as in the proposition. Let us suppose $[\mathbf{M}, \mathbf{A}] = 0$. Hence

$$\mathbf{m}_1 = 0 = \mathbf{m}_2, \quad m_{11} = m_{22} \quad \text{and} \quad m_{12} + m_{21} = 0.$$

Now if $\mathbf{M} \in \mathfrak{g}^e$, then $m_{11} = m_{22}$ and if $\mathbf{M} \in \mathfrak{g}^o$, then $m_{12} + m_{21} = 0$, which indeed means that $m_{12} = -m_{21} = t \in \mathbb{R}$. This shows that $\ker(ad_{\mathbf{A}\lambda})$ is as in the proposition. Also it is clear that the sum of the elements of the form of coefficients of λ^{2i+1} in $\ker(ad_{\mathbf{A}\lambda})$ and $\text{im}(ad_{\mathbf{A}\lambda})$ constitute \mathfrak{g}^o and those of λ^{2i} form \mathfrak{g}^e . This shows that the decomposition holds. \blacksquare

Notation 5.2.1. From now on, we denote the set of pure quaternionic numbers by $\text{im}(\mathbb{H})$.

Let us take Lax operator $L = D_x + \mathbf{A}\lambda + q$ as in (5.1.8), in which

$$q = \begin{pmatrix} 0 & 0 & \mathbf{0}^T \\ 0 & u & -\bar{\mathbf{u}}^T \\ \mathbf{0} & \mathbf{u} & \mathbf{0} \end{pmatrix} \in \mathfrak{sp}(n) \times \mathfrak{sp}(1).$$

Notice that in the case of real numbers instead of quaternions, we see that q is nothing but natural frame in comparison with the Frenêt frame in classical differential geometry.

Before we proceed, it would be useful to compute the bracket of general element $\mathbf{M} \in \mathfrak{b} = \mathfrak{sp}(n+1)$ as in (5.2.1) and the matrix $q \in \mathfrak{b}$ as we do it as below.

$$[M, U] = \begin{pmatrix} 0 & m_{12}u - \langle \mathbf{m}_1, \mathbf{u} \rangle & -m_{12}\bar{\mathbf{u}}^T \\ -um_{21} + \langle \mathbf{u}, \mathbf{m}_1 \rangle & C_{\mathbf{u}}\mathbf{m}_2 - C_u m_{22} & -m_{22}\bar{\mathbf{u}}^T + u\bar{\mathbf{m}}_2^T + \bar{\mathbf{u}}^T \mathbf{M} \\ -\mathbf{u}m_{21} & \mathbf{m}_2 u + \mathbf{M}\mathbf{u} - \mathbf{u}m_{22} & -\mathbf{m}_2\bar{\mathbf{u}}^T + \mathbf{u}\bar{\mathbf{m}}_2^T \end{pmatrix}.$$

where the inner product involved is the hermitian inner product, as we defined it Chapter 1.

Notation 5.2.2. The standard inner product on \mathbb{R}^n is denoted by \langle, \rangle_r .

Now the degree of σ is 2. Therefore the sequence of flows associated with Lax operators in (5.1.15) becomes

$$L_{t_{n+2}} = [A_{A\lambda^{n+2}}^+, L] = \lambda^2 L_{t_n} + [R_2, L], \quad (5.2.3)$$

and R_2 becomes

$$R_2 = N\lambda^2 + M\lambda + K, \quad N, K \in \mathfrak{g}^e \quad \text{and} \quad M \in \mathfrak{g}^o. \quad (5.2.4)$$

Now we compute coefficients of λ^3 of (5.2.3). First step: the coefficient of λ^3 . Simply the coefficient of λ^3 vanishes, that is :

$$[N, A] = 0. \quad (5.2.5)$$

This leads to following integrability conditions.

$$n_{11} = n_{22}, \quad (5.2.6a)$$

$$\mathbf{n}_2 = 0. \quad (5.2.6b)$$

The next coefficient is of λ^2 which gives us

$$0 = U_{t_m} + [N, U] + [M, A] - D_x N. \quad (5.2.7)$$

We derive the following equations using grading and equations above.

$$0 = m_{12} + m_{21} + D_x n_{11}, \quad (5.2.8a)$$

$$0 = m_{21} + m_{12} + n_{11}u - un_{11} - D_x n_{11} + u_{t_m}, \quad (5.2.8b)$$

$$0 = -\mathbf{m}_1 + \mathbf{u}n_{11} - \mathbf{u}_{t_m}, \quad (5.2.8c)$$

$$0 = D_x \mathbf{N}. \quad (5.2.8d)$$

From (5.2.8a) and (5.2.8b), we find that $(2D_x + C_u)n_{11} - u_{t_m} = 0$. Hence

$$n_{11} = (2D_x + C_u)^{-1}u_{t_m}.$$

As a result we obtain

$$\begin{aligned} \begin{pmatrix} m_{12} + m_{21} \\ \mathbf{m}_1 \end{pmatrix} &= \begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} Dn_{11} \\ \mathbf{m}_1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} D(C_u + 2D)^{-1} & 0 \\ L_{\mathbf{u}}(C_u + 2D)^{-1} & -I \end{pmatrix} \begin{pmatrix} u_{t_m} \\ \mathbf{u}_{t_m} \end{pmatrix} \\ &= \begin{pmatrix} -D(C_u + 2D)^{-1} & 0 \\ L_{\mathbf{u}}(C_u + 2D)^{-1} & -I \end{pmatrix} \begin{pmatrix} u_{t_m} \\ \mathbf{u}_{t_m} \end{pmatrix}. \end{aligned}$$

Denoting by \mathfrak{A} the operator

$$\mathfrak{A} =: \begin{pmatrix} -D(C_u + 2D)^{-1} & 0 \\ L_{\mathbf{u}}(C_u + 2D)^{-1} & -I \end{pmatrix},$$

we simply express last matrix equation as follows:

$$\begin{pmatrix} m_{12} + m_{21} \\ \mathbf{m}_1 \end{pmatrix} = \mathfrak{A} \begin{pmatrix} u_{t_m} \\ \mathbf{u}_{t_m} \end{pmatrix}.$$

Now the coefficient of λ is:

$$[M, U] + [K, A] - D_x M = 0. \quad (5.2.9)$$

Hence we derive the following equations:

$$0 = \mathbf{k}_2 + \mathbf{u}m_{21} + D_x \mathbf{m}_1, \quad (5.2.10a)$$

$$0 = D_x \mathbf{M}, \quad (5.2.10b)$$

$$0 = m_{12}u - Dm_{12} - k_{22} + k_{11} - \langle \mathbf{m}_1, \mathbf{u} \rangle, \quad (5.2.10c)$$

$$0 = -um_{21} - Dm_{21} - k_{22} + k_{11} + \langle \mathbf{u}, \mathbf{m}_1 \rangle = 0. \quad (5.2.10d)$$

By adding (5.2.10d) and (5.2.10c) we find that

$$\begin{aligned} 2k_{22} - 2k_{11} &= m_{12}u - um_{21} - D_x(m_{12} + m_{21}) + C_{\mathbf{u}}\mathbf{m}_1 \\ &= -\frac{1}{2}C_u(m_{12} + m_{21}) + u(m_{12} - m_{21}) \\ &\quad - D_x(m_{12} + m_{21}) + C_{\mathbf{u}}\mathbf{m}_1. \end{aligned}$$

Notice that we have used the fact that

$$m_{12}u - um_{21} = -\frac{1}{2}C_u(m_{12} + m_{21}) + u(m_{12} - m_{21}).$$

Also by subtracting those two equations we get that

$$m_{12}u + um_{21} - D(m_{12} - m_{21}) - A_{\mathbf{u}}\mathbf{m}_1 = 0.$$

On the other hand, $\frac{1}{2}A_u(m_{12} + m_{21}) = (m_{12}u + um_{21})$. Hence we obtain $m_{12} - m_{21}$ as follows:

$$m_{12} - m_{21} = D^{-1}\left(\frac{1}{2}A_u(m_{12} + m_{21}) - A_{\mathbf{u}}\mathbf{m}_1\right).$$

Remark 5.2.2. We can express this using the Killing form:

$$m_{12} - m_{21} = -\frac{1}{2}D^{-1}K\left(\begin{pmatrix} u & -\bar{\mathbf{u}}^T \\ \mathbf{u} & 0 \end{pmatrix}, \begin{pmatrix} m_{12} + m_{21} & -\bar{\mathbf{m}}_1^T \\ \mathbf{m}_1 & 0 \end{pmatrix}\right).$$

Also we can simply compute m_{21} as below.

$$\begin{aligned} m_{21} &= \frac{m_{21} - m_{12}}{2} + \frac{m_{21} + m_{12}}{2} \\ &= -\frac{1}{2}D^{-1}\left(\frac{1}{2}A_u(m_{12} + m_{21}) - A_{\mathbf{u}}\mathbf{m}_1\right) + \frac{m_{21} + m_{12}}{2}. \end{aligned}$$

We are able now to compute the following vector.

$$\begin{aligned} &\begin{pmatrix} k_{22} - k_{11} \\ \mathbf{k}_2 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{4}C_u + \frac{1}{2}uD^{-1}\frac{1}{2}A_u - \frac{1}{2}D & -\frac{1}{2}uD^{-1}A_{\mathbf{u}} + \frac{1}{2}C_{\mathbf{u}} \\ \frac{1}{2}L_{\mathbf{u}}D^{-1}(\frac{1}{2}A_u) - \frac{1}{2}L_{\mathbf{u}} & -\frac{1}{2}L_{\mathbf{u}}D^{-1}A_{\mathbf{u}} - D \end{pmatrix} \begin{pmatrix} m_{12} + m_{21} \\ \mathbf{m}_1 \end{pmatrix}. \end{aligned}$$

Let us denote the last operator we derived by \mathfrak{J} :

$$\mathfrak{J} = \begin{pmatrix} -\frac{1}{4}C_u + \frac{1}{2}uD^{-1}\frac{1}{2}A_u - \frac{1}{2}D & -\frac{1}{2}uD^{-1}A_{\mathbf{u}} + \frac{1}{2}C_{\mathbf{u}} \\ \frac{1}{2}L_{\mathbf{u}}D^{-1}(\frac{1}{2}A_u) - \frac{1}{2}L_{\mathbf{u}} & -\frac{1}{2}L_{\mathbf{u}}D^{-1}A_{\mathbf{u}} - D \end{pmatrix}.$$

Hence we write

$$\begin{pmatrix} k_{22} - k_{11} \\ \mathbf{k}_2 \end{pmatrix} = \mathfrak{J} \begin{pmatrix} m_{12} + m_{21} \\ \mathbf{m}_1 \end{pmatrix}. \quad (5.2.11)$$

The constant part gives

$$[K, U] - D_x K - U_{t_{m+2}} = 0, \quad (5.2.12)$$

from which one can find the following equation.

$$u_{t_{m+2}} = -(C_u + D)k_{22} + C_u \mathbf{k}_2, \quad (5.2.13a)$$

$$\mathbf{u}_{t_{m+2}} = -\mathbf{u}k_{22} + \mathbf{K}\mathbf{u} + \mathbf{k}_2 u - D\mathbf{k}_2, \quad (5.2.13b)$$

$$k_{11} = 0, \quad (5.2.13c)$$

$$DK^{ij} = \mathbf{u}^i \bar{\mathbf{k}}_2^j - \mathbf{k}_2^i \bar{\mathbf{u}}^j. \quad (5.2.13d)$$

This yields the expression for $\begin{pmatrix} u_{t_{m+2}} \\ \mathbf{u}_{t_{m+2}} \end{pmatrix}$ in terms of $\begin{pmatrix} k_{22} \\ \mathbf{k}_2 \end{pmatrix}$ as follows.

$$\begin{pmatrix} u_{t_{m+2}} \\ \mathbf{u}_{t_{m+2}} \end{pmatrix} = \begin{pmatrix} -(C_u + D) & C_u \\ -L_u & -D + R_u + \mathfrak{H}_1 \end{pmatrix} \begin{pmatrix} k_{22} \\ \mathbf{k}_2 \end{pmatrix}, \quad (5.2.14)$$

in which the notation \mathfrak{H}_1 is used to be the operator acting as follows:

$$\mathfrak{H}_1 \mathbf{k}_2 = D^{-1}(\mathbf{u}^i \bar{\mathbf{k}}_2^j - \mathbf{k}_2^i \bar{\mathbf{u}}^j) \mathbf{u}.$$

We denote by \mathfrak{H} the operator that just appeared, i.e.,

$$\mathfrak{H} =: \begin{pmatrix} -(C_u + D) & C_u \\ -L_u & -D + R_u + \mathfrak{H}_1 \end{pmatrix}.$$

Hence the constant coefficient of Lax representation can be written as follows.

$$\begin{pmatrix} u_{t_{m+2}} \\ \mathbf{u}_{t_{m+2}} \end{pmatrix} = \mathfrak{H} \mathfrak{A} \begin{pmatrix} u_{t_m} \\ \mathbf{u}_{t_m} \end{pmatrix}.$$

There is a link between the operators \mathfrak{A} and \mathfrak{H} in following proposition.

Proposition 5.2.3.

$$\mathfrak{A}^{-1} \mathfrak{H} \mathfrak{A}^* = \mathfrak{H}.$$

Proof. See Proposition 4.3.6 of Chapter 4. ■

Remark 5.2.4. This may be related to the Hamiltonian map as in [40, theorem 5.2.9], the map Φ is Hamiltonian if and only if

$$\Phi(B) = D(\Phi)B_1D(\Phi)^*,$$

where B and B_1 are Hamiltonian operator acting on C and C_1 , respectively.

Remark 5.2.5. We can see that:

$$\begin{aligned}
& (D_x + L_u)(2D_x + C_u)^{-1}(-D_x + R_u) = \\
&= \left(D_x + \frac{A_u + C_u}{2}\right)(2D_x + C_u)^{-1}\left(-D_x + \frac{A_u - C_u}{2}\right) \\
&= \left(\frac{A_u}{2} + \frac{2D_x + C_u}{2}\right)(2D_x + C_u)^{-1}\left(\frac{A_u}{2} - \frac{2D_x + C_u}{2}\right) \\
&= -\frac{1}{2}D_x + \frac{1}{2}A_u(2D_x + C_u)^{-1}\frac{1}{2}A_u - \frac{1}{4}C_u \\
&= -\frac{1}{2}D_x + \frac{1}{2}uD_x^{-1}\frac{1}{2}A_u - \frac{1}{4}C_u.
\end{aligned}$$

Therefore we can write the operator \mathfrak{J} as below.

$$\mathfrak{J} = \begin{pmatrix} (D_x + L_u)(2D_x + C_u)^{-1}(-D_x + R_u) & -\frac{1}{2}uD_x^{-1}A_u + \frac{1}{2}C_u \\ \frac{1}{2}L_u D^{-1}(\frac{1}{2}A_u) - \frac{1}{2}L_u & -\frac{1}{2}L_u D^{-1}A_u - D \end{pmatrix}.$$

See Remark 4.3.14 for the motivation of this decomposition.

Now if we let $u_{t_m} = u_1$ and $\mathbf{u}_{t_m} = \mathbf{u}_1$, as trivial symmetry, then we get the following system of scalar-vector equation.

$$\begin{cases} u_t = -\frac{1}{4}u_3 + \frac{3}{8}(uu_1u - uu_2 + u_2u) - \frac{3}{2} \langle \mathbf{u}, \mathbf{u} \rangle u_1 \\ -2 \langle \mathbf{u}, \mathbf{u}_1 \rangle u + \frac{1}{2}u \langle \mathbf{u}, \mathbf{u}_1 \rangle - u \langle \mathbf{u}_1, \mathbf{u} \rangle \\ -\frac{1}{2} \langle \mathbf{u}_1, \mathbf{u} \rangle u + \frac{3}{2}C_u \mathbf{u}_2, \\ \mathbf{u}_t = -\mathbf{u}_3 + \frac{3}{2}\mathbf{u}_2u + \mathbf{u}_1\left(\frac{3}{4}u_1 - \frac{3}{8}u^2 - \frac{3}{2} \langle \mathbf{u}, \mathbf{u} \rangle\right). \end{cases} \quad (5.2.15)$$

We check that if we apply again

$$\mathfrak{R} = \mathfrak{H}\mathfrak{J}\mathfrak{A},$$

on the equation itself, the result will commute with the equation itself. This means that we can construct the hierarchy of equations starting with this new integrable system. The splitting the operator \mathfrak{R} to Hamiltonian and symplectic has been worked out in Chapter 4.

Remark 5.2.6. We compute the previous equation (5.2.15), explicitly and step by step. Let $u_{t_m} = u_1$ and $\mathbf{u}_{t_m} = \mathbf{u}_1$ then

$$\begin{pmatrix} Dn_{11} \\ \mathbf{m}_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}u_1 \\ \frac{1}{2}\mathbf{u}u - \mathbf{u}_1 \end{pmatrix}.$$

Hence

$$\begin{pmatrix} m_{12} + m_{21} \\ \mathbf{m}_1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}u_1 \\ \frac{1}{2}\mathbf{u}u - \mathbf{u}_1 \end{pmatrix}.$$

Thus we obtain $\begin{pmatrix} k_{22} \\ \mathbf{k}_2 \end{pmatrix}$ according to (5.2.11) as follows.

$$\begin{aligned} \begin{pmatrix} k_{22} \\ \mathbf{k}_2 \end{pmatrix} &= \begin{pmatrix} \frac{1}{8}C_u u_1 + \frac{1}{4}u_2 + (\frac{1}{2}u \langle \mathbf{u}, \mathbf{u} \rangle - \frac{1}{2}C_{\mathbf{u}} \mathbf{u}_1) + \frac{1}{2}u(-\frac{1}{4}u^2 + \langle \mathbf{u}, \mathbf{u} \rangle) \\ \frac{1}{2}\mathbf{u}(-\frac{1}{8}u^2 + \langle \mathbf{u}, \mathbf{u} \rangle) + \frac{1}{4}\mathbf{u}u_1 - (\frac{1}{2}\mathbf{u}_1 u + \frac{1}{2}\mathbf{u}u_1 - \mathbf{u}_2) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{8}C_u u_1 + \frac{1}{4}u_2 + (u \langle \mathbf{u}, \mathbf{u} \rangle - \frac{1}{2}C_{\mathbf{u}} \mathbf{u}_1) - \frac{1}{8}u^3 \\ \frac{1}{2}\mathbf{u}(-\frac{1}{8}u^2 + \langle \mathbf{u}, \mathbf{u} \rangle) - (\frac{1}{2}\mathbf{u}_1 u + \frac{1}{4}\mathbf{u}u_1 - \mathbf{u}_2) \end{pmatrix}, \end{aligned}$$

and so (5.2.14) yields the following expression for $\begin{pmatrix} u_{t_{m+2}} \\ \mathbf{u}_{t_{m+2}} \end{pmatrix}$ explicitly in terms of $\begin{pmatrix} u \\ \mathbf{u} \end{pmatrix}$ and its total derivatives.

$$\begin{pmatrix} u_{t_{m+2}} \\ \mathbf{u}_{t_{m+2}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{4}u_3 + \frac{3}{8}(uu_1 u - uu_2 + u_2 u) - \frac{3}{2} \langle \mathbf{u}, \mathbf{u} \rangle u_1 \\ -2 \langle \mathbf{u}, \mathbf{u}_1 \rangle u + \frac{1}{2}u \langle \mathbf{u}, \mathbf{u}_1 \rangle - u \langle \mathbf{u}_1, \mathbf{u} \rangle \\ -\frac{1}{2} \langle \mathbf{u}_1, \mathbf{u} \rangle u + \frac{3}{2}C_{\mathbf{u}} \mathbf{u}_2 \\ -\mathbf{u}_3 + \frac{3}{2}\mathbf{u}_2 u + \mathbf{u}_1(\frac{3}{4}u_1 - \frac{3}{8}u^2 - \frac{3}{2} \langle \mathbf{u}, \mathbf{u} \rangle) \end{pmatrix}.$$

5.3 Higher symmetry

We apply the recursion operator \mathfrak{R} to the equation 5.2.15 and find fifth order equation of the hierarchy as follows To have short expression, let us put $\mathfrak{R} \begin{pmatrix} u \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} S_0 \\ \mathbf{S}_1 \end{pmatrix}$. Then we have that

$$\begin{aligned} \mathbf{S} &= 16\mathbf{u}_5 + (-40u)\mathbf{u}_4 + \mathbf{u}_3 \left[30u^2 - 60u_1 + 40 \langle \mathbf{u}, \mathbf{u} \rangle \right] \\ &+ \mathbf{u}_2 \left[60 \langle \mathbf{u}_1, \mathbf{u} \rangle + 20 \langle \mathbf{u}, \mathbf{u}_1 \rangle - 24 \langle \mathbf{u}, \mathbf{u} \rangle u - 50u_2 \right. \\ &\left. + 10u_1 u + 10u_1 u_0 - 36u \langle \mathbf{u}, \mathbf{u} \rangle + 50uu_1 - 5u^3 \right] \\ &+ \mathbf{u}_1 \left[50 \langle \mathbf{u}_2, \mathbf{u} \rangle + 40 \langle \mathbf{u}_1, \mathbf{u}_1 \rangle - 7 \langle \mathbf{u}_1, \mathbf{u} \rangle u - 7 \langle \mathbf{u}_1, \mathbf{u} \rangle u \right. \\ &\left. + 30 \langle \mathbf{u}, \mathbf{u}_2 \rangle + 23 \langle \mathbf{u}, \mathbf{u}_1 \rangle u + 30 \langle \mathbf{u}, \mathbf{u} \rangle^2 - 15u_3 + \frac{5}{2}u_2 u \right. \\ &\left. - 30u_1 \langle \mathbf{u}, \mathbf{u} \rangle - \frac{5}{2}uu_1 u + \frac{45}{2}uu_2 - 23u \langle \mathbf{u}, \mathbf{u}_1 \rangle \right. \\ &\left. - 53u \langle \mathbf{u}_1, \mathbf{u} \rangle + \frac{25}{2}u_1 u^2 u_1^2 - \frac{5}{8}u^4 - 6u^2 u_1 + 15u^2 \langle \mathbf{u}, \mathbf{u} \rangle \right. \\ &\left. - \frac{5}{2}uu_1 u - 23u \langle \mathbf{u}, \mathbf{u}_1 \rangle - 53u \langle \mathbf{u}_1, \mathbf{u} \rangle + u_1 u^2 + \frac{25}{2}u_1^2 \right]. \end{aligned}$$

and also we obtain that

$$\begin{aligned}
S_0 = & u_5 + \frac{5}{2}u_4u - \frac{5}{2}uu_4 - 5uu_3u + u^2u_3 + \frac{5}{2}u_3u_1 + \frac{3}{2}u_3u^2 - \frac{5}{2}u_1u_3 \\
& \frac{1}{2}u_2u_1u - u_2uu_1 - \frac{3}{8}u_2u^3 - \frac{7}{2}u_1u_2u - \frac{3}{2}u_1uu_2 - 4uu_2u_1 \\
& - \frac{19}{8}uu_2u^2 - \frac{1}{2}uu_1u_2 + \frac{13}{8}u^2u_2u + \frac{9}{8}u^3u_2 \\
& - \frac{1}{4}u^4u_1 + \frac{7}{8}u^3u_1u + \frac{7}{8}u^2u_1u^2 + \frac{7}{4}u^2u_1^2 + \frac{3}{8}uu_1u^3 + \frac{5}{4}uu_1uu_1 \\
& - \frac{3}{4}uu_1^2u + \frac{1}{4}u_1u^2u_1 - \frac{5}{4}u_1uu_1u - \frac{5}{4}u_1^2u^2 - \frac{5}{2}u_1^3 \\
& 3u^4 \langle \mathbf{u}, \mathbf{u} \rangle - 3u^3 \langle \mathbf{u}, \mathbf{u} \rangle u - 7u^3 \langle \mathbf{u}, \mathbf{u}_1 \rangle - 2u^3 \langle \mathbf{u}_1, \mathbf{u} \rangle \\
& - 6u^2 \langle \mathbf{u}, \mathbf{u} \rangle^2 + 5u^2 \langle \mathbf{u}, \mathbf{u}_1 \rangle u + 3u^2 \langle \mathbf{u}, \mathbf{u}_2 \rangle + 5u^2 \langle \mathbf{u}_1, \mathbf{u} \rangle u \\
& + 6u^2 \langle \mathbf{u}_1, \mathbf{u}_1 \rangle - 27u^2 \langle \mathbf{u}_2, \mathbf{u} \rangle u + 6u \langle \mathbf{u}, \mathbf{u} \rangle^2 u \\
& + 12u \langle \mathbf{u}, \mathbf{u} \rangle \langle \mathbf{u}, \mathbf{u}_1 \rangle + 24u \langle \mathbf{u}, \mathbf{u} \rangle \langle \mathbf{u}_1, \mathbf{u} \rangle \\
& - 7u \langle \mathbf{u}, \mathbf{u}_1 \rangle u^2 - 25u \langle \mathbf{u}, \mathbf{u}_2 \rangle u - 13u \langle \mathbf{u}, \mathbf{u}_3 \rangle \\
& - 7u \langle \mathbf{u}_1, \mathbf{u} \rangle u^2 + 12u \langle \mathbf{u}_1, \mathbf{u} \rangle \langle \mathbf{u}, \mathbf{u} \rangle - 20 \langle \mathbf{u}_1, \mathbf{u}_1 \rangle u \\
& - 9u \langle \mathbf{u}_1, \mathbf{u}_2 \rangle + 5u \langle \mathbf{u}_2, \mathbf{u} \rangle u + 21u \langle \mathbf{u}_2, \mathbf{u}_1 \rangle \\
& + 57u \langle \mathbf{u}_3, \mathbf{u}_1 \rangle - 8 \langle \mathbf{u}, \mathbf{u} \rangle u \langle \mathbf{u}, \mathbf{u}_1 \rangle + 16 \langle \mathbf{u}, \mathbf{u} \rangle u \langle \mathbf{u}_1, \mathbf{u} \rangle \\
& + 26 \langle \mathbf{u}, \mathbf{u} \rangle \langle \mathbf{u}, \mathbf{u}_1 \rangle + 2 \langle \mathbf{u}, \mathbf{u} \rangle \langle \mathbf{u}_1, \mathbf{u} \rangle u + 4 \langle \mathbf{u}, \mathbf{u}_1 \rangle u^3 \\
& + 24 \langle \mathbf{u}, \mathbf{u}_1 \rangle u \langle \mathbf{u}, \mathbf{u} \rangle + 9 \langle \mathbf{u}_2, \mathbf{u}_1 \rangle u - 7 \langle \mathbf{u}_3, \mathbf{u} \rangle u \\
& - 6 \langle \mathbf{u}_1, \mathbf{u} \rangle \langle \mathbf{u}, \mathbf{u} \rangle u + 14 \langle \mathbf{u}_1, \mathbf{u}_1 \rangle u^2 + 63 \langle \mathbf{u}, \mathbf{u}_3 \rangle u \\
& - \langle \mathbf{u}_1, \mathbf{u} \rangle u^3 + 18 \langle \mathbf{u}, \mathbf{u}_1 \rangle \langle \mathbf{u}, \mathbf{u} \rangle u + 37 \langle \mathbf{u}, \mathbf{u}_2 \rangle u^2 \\
& + 12 \langle \mathbf{u}_1, \mathbf{u} \rangle u \langle \mathbf{u}, \mathbf{u} \rangle + 7 \langle \mathbf{u}_2, \mathbf{u} \rangle u^2 + 39 \langle \mathbf{u}_1, \mathbf{u}_2 \rangle u \\
& - 6uu_1u \langle \mathbf{u}, \mathbf{u} \rangle - 3uu_1 \langle \mathbf{u}, \mathbf{u} \rangle u - 4uu_1 \langle \mathbf{u}, \mathbf{u}_1 \rangle - 24uu_1 \langle \mathbf{u}_1, \mathbf{u} \rangle \\
& - 14u \langle \mathbf{u}, \mathbf{u}_1 \rangle u_1 - 12u \langle \mathbf{u}, \mathbf{u}_1 \rangle \langle \mathbf{u}, \mathbf{u} \rangle - 14u \langle \mathbf{u}_1, \mathbf{u} \rangle u_1 \\
& - 3u_1u^2 \langle \mathbf{u}, \mathbf{u} \rangle + 3u_1u \langle \mathbf{u}, \mathbf{u} \rangle u + 7u_1u \langle \mathbf{u}, \mathbf{u}_1 \rangle - 3u_1u \langle \mathbf{u}_1, \mathbf{u} \rangle \\
& + 4u_1 \langle \mathbf{u}_1, \mathbf{u} \rangle u + 12u_1 \langle \mathbf{u}_1, \mathbf{u}_1 \rangle + 66u_1 \langle \mathbf{u}_2, \mathbf{u} \rangle + 6u_1 \langle \mathbf{u}, \mathbf{u} \rangle^2 \\
& + 4u_1 \langle \mathbf{u}, \mathbf{u}_1 \rangle u + 6u_1 \langle \mathbf{u}, \mathbf{u}_2 \rangle - 6 \langle \mathbf{u}, \mathbf{u} \rangle uu_1u + 8 \langle \mathbf{u}, \mathbf{u}_1 \rangle uu_1 \\
& + 24 \langle \mathbf{u}, \mathbf{u} \rangle u_1 \langle \mathbf{u}, \mathbf{u} \rangle + 29 \langle \mathbf{u}, \mathbf{u}_1 \rangle u_1u + 74 \langle \mathbf{u}, \mathbf{u}_2 \rangle u_1 \\
& - 2 \langle \mathbf{u}_1, \mathbf{u} \rangle uu_1 + 9 \langle \mathbf{u}_1, \mathbf{u} \rangle u_1u + 28 \langle \mathbf{u}_1, \mathbf{u}_1 \rangle u_1 + 14 \langle \mathbf{u}_2, \mathbf{u} \rangle u_1 \\
& - 9uu_2 \langle \mathbf{u}, \mathbf{u} \rangle + 3u_2u \langle \mathbf{u}, \mathbf{u} \rangle + 6u_2 \langle \mathbf{u}, \mathbf{u} \rangle u + 11u_2 \langle \mathbf{u}, \mathbf{u}_1 \rangle \\
& + 31u_2 \langle \mathbf{u}_1, \mathbf{u} \rangle - 6 \langle \mathbf{u}, \mathbf{u} \rangle uu_2 + 6 \langle \mathbf{u}, \mathbf{u} \rangle u_2u + 29 \langle \mathbf{u}, \mathbf{u}_1 \rangle u_2 \\
& + 9 \langle \mathbf{u}_1, \mathbf{u} \rangle u_2 + 6u_3 \langle \mathbf{u}, \mathbf{u} \rangle + 4 \langle \mathbf{u}, \mathbf{u} \rangle u_3 \\
& + 18 \langle \mathbf{u}, \mathbf{u} \rangle \langle \mathbf{u}, \mathbf{u}_2 \rangle - 12 \langle \mathbf{u}, \mathbf{u} \rangle \langle \mathbf{u}_1, \mathbf{u}_1 \rangle \\
& - 54 \langle \mathbf{u}, \mathbf{u} \rangle \langle \mathbf{u}_2, \mathbf{u} \rangle - 20 \langle \mathbf{u}_3, \mathbf{u}_1 \rangle - 30 \langle \mathbf{u}_4, \mathbf{u} \rangle + 30 \langle \mathbf{u}, \mathbf{u}_4 \rangle \\
& - 6 \langle \mathbf{u}_2, \mathbf{u} \rangle \langle \mathbf{u}, \mathbf{u} \rangle + 12 \langle \mathbf{u}_1, \mathbf{u}_1 \rangle \langle \mathbf{u}, \mathbf{u} \rangle \\
& + 42 \langle \mathbf{u}, \mathbf{u}_2 \rangle \langle \mathbf{u}, \mathbf{u} \rangle + 60 \langle \mathbf{u}, \mathbf{u}_1 \rangle^2 + 20 \langle \mathbf{u}_1, \mathbf{u}_3 \rangle - 60 \langle \mathbf{u}_1, \mathbf{u} \rangle^3,
\end{aligned}$$

The code we used to find out that the equation itself will commute with what we can find by applying the recursion operator is written in FORM, see [72] and [57].

