Integrable Systems and Symplectic Geometry

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Chapter 6

Computation of Geometric operator

6.1 Hamiltonian operator

This section is devoted to the computation of the Hamiltonian operator. In Chapter 4, Theorem 4.3.12, we claimed that the operator $H = \mathfrak{H}\mathfrak{A}^*$ is indeed Hamiltonian. We proved also that the operator $\mathfrak{A}$ is a Nijenhuis operator, see Theorem 4.3.8. As is known, see for instance [69], the operator $\mathfrak{H}$ is indeed a Hamiltonian operator. We will prove that the Lie algebra form of $\mathfrak{H}$ which is denoted by $\hat{\mathfrak{H}}$, as in Theorem 4.4.3, is Hamiltonian using the definition and techniques described in Chapter 2. We will prove that

$$\int K(\hat{p}_i, D_{\hat{\mathfrak{H}}}[\hat{\mathfrak{H}}\hat{p}_{i+2}](\hat{p}_{i+1})) = 0. \tag{6.1.1}$$

Notation 6.1.1. Here we have taken the sum over index i, but we take into account the rule of shifting, that is for instance, we can add to index i, by $1, 2$ and use, for instance, the fact that

$$\hat{p}_{i+3} = \hat{p}_i, \quad \hat{p}_{i+4} = \hat{p}_{i+1}. \tag{6.1.2}$$

Notation 6.1.2. Here and after, we simply use the notation H for the operator $\hat{\mathfrak{H}}$ and p_i for the matrix $\hat{p}_i$ and likewise for $\hat{u}$ so that we write as

$$H = D_x - \pi_1 \mathrm{ad}_u - \mathrm{ad}_u D^{-1} \pi_0 \mathrm{ad}_u.$$

The Fréchet derivative of H is

$$D_H[q] = -\pi_1 \mathrm{ad}_q - \mathrm{ad}_q D^{-1} \pi_0 \mathrm{ad}_u - \mathrm{ad}_u D^{-1} \pi_0 \mathrm{ad}_q.$$

Now we compute the expression on the left of (6.1.1):

$$\begin{aligned} & \int K(p_i, D_H[Hp_{i+2}](p_{i+1})) \\ = & \int K\Big(p_i, (-\pi_1 \mathrm{ad}_{Hp_{i+2}} - \mathrm{ad}_{Hp_{i+2}} D^{-1}\pi_0 \mathrm{ad}_u - \mathrm{ad}_u D^{-1} \pi_0 \mathrm{ad}_{Hp_{i+2}}) p_{i+1}\Big) \\ = & -\int K(p_i, \pi_1 \mathrm{ad}_{p_{i+2,x} - \pi_1 \mathrm{ad}_u p_{i+2} - \mathrm{ad}_u D_x^{-1} \pi_0 \mathrm{ad}_u p_{i+2}} p_{i+1}) \\ & -\int K(p_i, \mathrm{ad}_{p_{i+2,x} - \pi_1 \mathrm{ad}_u p_{i+2} - \mathrm{ad}_u D_x^{-1} \pi_0 \mathrm{ad}_u p_{i+2}} D^{-1} \pi_0 \mathrm{ad}_u p_{i+1}) \\ & -\int K(p_i, \mathrm{ad}_u D^{-1} \pi_0 \mathrm{ad}_{p_{i+2,x} - \pi_1 \mathrm{ad}_u p_{i+2} - \mathrm{ad}_u D_x^{-1} \pi_0 \mathrm{ad}_u p_{i+2}} p_{i+1}). \end{aligned} \tag{6.1.3}$$

This expression is a combination of the Schouten brackets of three operators constituting the operator H, these are, $H_1 = D_x, H_2 = \pi_1 \mathrm{ad}_u$ and $H_3 = \mathrm{ad}_u D^{-1}\pi_0 \mathrm{ad}_u$. In order to prove that H is Hamiltonian, we show that

$$[H_i, H_j] = 0, \quad \text{for} \quad i,j = 1,2,3.$$

We will break the proof of this claim into the several part.

Lemma 6.1.1. The Schouten bracket $[H_3, H_3]$ vanishes.

Proof. We explain every single step and every single rule we use, so that later on we will just do it. It is clear from (6.1.3) that

$$\begin{aligned}[H_3, H_3](p_1, p_2, p_3) &= \int K(p_i, \mathrm{ad}_{\mathrm{ad}_u D_x^{-1}\pi_0 \mathrm{ad}_u p_{i+2}} D^{-1}\pi_0 \mathrm{ad}_u p_{i+1}) \\ &+ \int K(p_i, \mathrm{ad}_u D^{-1}\pi_0 \mathrm{ad}_{\mathrm{ad}_u D_x^{-1}\pi_0 \mathrm{ad}_u p_{i+2}} p_{i+1}).\end{aligned}$$

We simplify the first term as follows:

$$\begin{aligned}&\int K(p_i, \mathrm{ad}_{\mathrm{ad}_u D_x^{-1}\pi_0 \mathrm{ad}_u p_{i+2}} D^{-1}\pi_0 \mathrm{ad}_u p_{i+1}) \\ =& \int K(p_{i+1}, \mathrm{ad}_{\mathrm{ad}_u D_x^{-1}\pi_0 \mathrm{ad}_u p_i} D^{-1}\pi_0 \mathrm{ad}_u p_{i+2}).\end{aligned}$$

in which we have used the shifting rule (6.1.2). For the second term, we derive the following:

$$\begin{aligned}&\int K(p_i, \mathrm{ad}_u D^{-1}\pi_0 \mathrm{ad}_{\mathrm{ad}_u D_x^{-1}\pi_0 \mathrm{ad}_u p_{i+2}} p_{i+1}) \\ =& -\int K(\mathrm{ad}_u p_i, D^{-1}\pi_0 \mathrm{ad}_{\mathrm{ad}_u D_x^{-1}\pi_0 \mathrm{ad}_u p_{i+2}} p_{i+1}) \\ =& \int K(D^{-1}\mathrm{ad}_u p_i, \pi_0 \mathrm{ad}_{\mathrm{ad}_u D_x^{-1}\pi_0 \mathrm{ad}_u p_{i+2}} p_{i+1}) \\ =& \int K(D^{-1}\pi_0 \mathrm{ad}_u p_i, \mathrm{ad}_{\mathrm{ad}_u D_x^{-1}\pi_0 \mathrm{ad}_u p_{i+2}} p_{i+1}) \\ =& -\int K(\mathrm{ad}_{\mathrm{ad}_u D_x^{-1}\pi_0 \mathrm{ad}_u p_{i+2}} D^{-1}\pi_0 \mathrm{ad}_u p_i, p_{i+1}),\end{aligned}$$

where the first equality follows from the fact that the Killing form is invariant under the adjoint action, that is,

$$K(\mathrm{ad}_X Y, Z) + K(Y, \mathrm{ad}_X Z) = 0, \quad X, Y, X \in \mathfrak{g}. \tag{6.1.4}$$

The second equality follows the integration by parts, the third equality follows the rule stated in Lemma 1.4.2, the fourth equality again the invariance of the Killing form.

Hence we obtain

$$\begin{aligned}[H_3,H_3](p_1,p_2,p_3) &= \int K(p_{i+1}, \mathrm{ad}_{\mathrm{ad}_u D_x^{-1}\pi_0 \mathrm{ad}_u p_i} D^{-1}\pi_0 \mathrm{ad}_u p_{i+2}) \\ &- \int K(\mathrm{ad}_{\mathrm{ad}_u D_x^{-1}\pi_0 \mathrm{ad}_u p_{i+2}} D^{-1}\pi_0 \mathrm{ad}_u p_i, p_{i+1}).\end{aligned}$$

Now using the Jacobi identity for the three elements $u, D^{-1}\pi_0\mathrm{ad}_u p_{i+2}$ and $D^{-1}\pi_0\mathrm{ad}_u p_i$ of the Lie algebra $\mathfrak{g}$, we obtain

$$[H_3,H_3](p_1,p_2,p_3) = \int K(p_{i+1}, \mathrm{ad}_u \mathrm{ad}_{D_x^{-1}\pi_0 \mathrm{ad}_u p_i} D^{-1}\pi_0 \mathrm{ad}_u p_{i+2}).$$

Again using the rules mentioned above, we see that

$$\begin{aligned}[H_3,H_3](p_1,p_2,p_3) &= -\int K(\mathrm{ad}_u p_{i+1}, \mathrm{ad}_{D_x^{-1}\pi_0 \mathrm{ad}_u p_i} D^{-1}\pi_0 \mathrm{ad}_u p_{i+2}) \\ &= -\int K(\pi_0 \mathrm{ad}_u p_{i+1}, \mathrm{ad}_{D_x^{-1}\pi_0 \mathrm{ad}_u p_i} D_x^{-1}\pi_0 \mathrm{ad}_u p_{i+2}).\end{aligned}$$

The last integrand we obtained is in the image of total derivative, more precisely,

Remark 6.1.2.

$$\begin{aligned}D_x K(p, \mathrm{ad}_q r) &= K(p_x, \mathrm{ad}_q r) + K(p, \mathrm{ad}_{q_x} r) + K(p, \mathrm{ad}_q r_x) \\ &= K(p_x, \mathrm{ad}_q r) + K(\mathrm{ad}_r p, q_x) - K(\mathrm{ad}_q p, r_x).\end{aligned}$$

Hence

$$D_x K(p_i, \mathrm{ad}_{p_{i+1}} p_{i+2}) = 3K(p_{i_x}, \mathrm{ad}_{p_{i+1}} p_{i+2}).$$

Therefore $K(p_{i_x}, \mathrm{ad}_{p_{i+1}} p_{i+2}) = 1/3 D_x K(p_i, \mathrm{ad}_{p_{i+1}} p_{i+2})$.

It follows that

$$[H_3,H_3](p_1,p_2,p_3) = -\frac{1}{3}\int D_x K(D_x^{-1}\pi_0 \mathrm{ad}_u p_{i+1}, \mathrm{ad}_{D_x^{-1}\pi_0 \mathrm{ad}_u p_i} D_x^{-1}\pi_0 \mathrm{ad}_u p_{i+2}).$$

Hence by definition $[H_3,H_3](p_1,p_2,p_3) = 0$. ■

Lemma 6.1.3. $[H_3, H_2] = 0$.

Proof. From (6.1.3), we obtain that

$$\begin{aligned}&[H_3,H_2](p_1,p_2,p_3) \\ = \; &+\int K(p_i, \pi_1 \mathrm{ad}_{\mathrm{ad}_u D_x^{-1}\pi_0 \mathrm{ad}_u p_{i+2}} p_{i+1}) + \int K(p_i, \mathrm{ad}_{\pi_1 \mathrm{ad}_u p_{i+2}} D^{-1}\pi_0 \mathrm{ad}_u p_{i+1}) \\ &+\int K(p_i, \mathrm{ad}_u D^{-1}\pi_0 \mathrm{ad}_{\pi_1 \mathrm{ad}_u p_{i+2}} p_{i+1}). \qquad (6.1.5)\end{aligned}$$

Now we manipulate the expression on the right. The first term is simplified as follows:

$$\begin{aligned} & \int K(p_i\,, \pi_1 \mathrm{ad}_{\mathrm{ad}_u D_x^{-1}\pi_0 \mathrm{ad}_u p_{i+2}} p_{i+1}) = \int K(p_i\,, \mathrm{ad}_{\mathrm{ad}_u D_x^{-1}\pi_0 \mathrm{ad}_u p_{i+2}} p_{i+1}) \\ = & -\int K(\mathrm{ad}_{\mathrm{ad}_u D_x^{-1}\pi_0 \mathrm{ad}_u p_{i+2}} p_i\,, p_{i+1}) = \int K(\mathrm{ad}_{p_i} \mathrm{ad}_u D_x^{-1}\pi_0 \mathrm{ad}_u p_{i+2}\,, p_{i+1}) \\ = & -\int K(\mathrm{ad}_u D_x^{-1}\pi_0 \mathrm{ad}_u p_{i+2}\,, \mathrm{ad}_{p_i} p_{i+1}) = \int K(D_x^{-1}\pi_0 \mathrm{ad}_u p_{i+2}\,, \mathrm{ad}_u \mathrm{ad}_{p_i} p_{i+1}) \\ = & \int K(D_x^{-1}\pi_0 \mathrm{ad}_u p_{i+2}\,, \pi_0 \mathrm{ad}_u \mathrm{ad}_{p_i} p_{i+1}) \end{aligned}$$

Here we have used Lemma 1.4.2, 6.1.4, anti-symmetricity of the Lie bracket, (6.1.4) (twice), and Lemma 1.4.2, respectively.

The second and third terms of (6.1.5) together simplify as follows:

$$\begin{aligned} & \int K(p_i, \mathrm{ad}_{\pi_1 \mathrm{ad}_u p_{i+2}} D^{-1}\pi_0 \mathrm{ad}_u p_{i+1}) + K(p_i, \mathrm{ad}_u D_x^{-1}\pi_0 \mathrm{ad}_{\pi_1 \mathrm{ad}_u p_{i+2}} p_{i+1}) \\ = & -\int K(\mathrm{ad}_{\pi_1 \mathrm{ad}_u p_{i+2}} p_i, D_x^{-1}\pi_0 \mathrm{ad}_u p_{i+1}) - K(\mathrm{ad}_u p_i, D^{-1}\pi_0 \mathrm{ad}_{\pi_1 \mathrm{ad}_u p_{i+2}} p_{i+1}) \\ = & -\int K(\mathrm{ad}_{\pi_1 \mathrm{ad}_u p_{i+2}} p_i, D_x^{-1}\pi_0 \mathrm{ad}_u p_{i+1}) \\ + & \int K(D_x^{-1} \mathrm{ad}_u p_i, D^{-1}\pi_0 \mathrm{ad}_{\pi_1 \mathrm{ad}_u p_{i+2}} p_{i+1}) \\ = & -\int K(\pi_0 \mathrm{ad}_{\pi_1 \mathrm{ad}_u p_{i+2}} p_i, D^{-1}\pi_0 \mathrm{ad}_u p_{i+1}) \\ + & \int K(D_x^{-1}\pi_0 \mathrm{ad}_u p_{i+1}, \pi_0 \mathrm{ad}_{\pi_1 \mathrm{ad}_u p_i} p_{i+2}) \\ = & -\int K(\pi_0 \mathrm{ad}_{\mathrm{ad}_u p_{i+2}} p_i, D_x^{-1}\pi_0 \mathrm{ad}_u p_{i+1}) + K(D_x^{-1}\pi_0 \mathrm{ad}_u p_{i+1}, \pi_0 \mathrm{ad}_{\mathrm{ad}_u p_i} p_{i+2}) \\ = & -\int K(\pi_0(\mathrm{ad}_{\mathrm{ad}_u p_{i+2}} p_i - \mathrm{ad}_{\mathrm{ad}_u p_i} p_{i+2}), D_x^{-1}\pi_0 \mathrm{ad}_u p_{i+1}) \\ = & \int K(\pi_0 \mathrm{ad}_u \mathrm{ad}_{p_i} p_{i+2}, D_x^{-1}\pi_0 \mathrm{ad}_u p_{i+1}) \\ = & \int K(\pi_0 \mathrm{ad}_u \mathrm{ad}_{p_{i+1}} p_i, D_x^{-1}\pi_0 \mathrm{ad}_u p_{i+2}). \end{aligned}$$

Here the first is term modified according to (6.1.4), Lemma 1.4.2, and finally the fact that one has

$$\pi_0 \mathrm{ad}_{\pi_1 \mathrm{ad}_u p_{i+2}} p_i = \pi_0 \mathrm{ad}_{\mathrm{ad}_u p_{i+2}} p_i. \tag{6.1.6}$$

Similarly the second term is manipulated using (6.1.4), integration by part, Lemma 1.4.2, and again the equality (6.1.6). Then these two terms together modified using the Jacobi identity and then shifting rule. Now it should be clear that

$$[H_3, H_2](p_1, p_2, p_3)$$

vanishes. ■

Lemma 6.1.4. $[H_2, H_2] = 0$.

Proof. We see that

$$\begin{aligned}
[H_2, H_2](p_1, p_2, p_3) &= \int K(p_i, \pi_1 \mathrm{ad}_{\pi_1 \mathrm{ad}_u p_{i+2}} p_{i+1}) \\
&= \int K(p_i, \mathrm{ad}_{\pi_1 \mathrm{ad}_u p_{i+2}} p_{i+1}) \\
&= -\int K(p_i, \mathrm{ad}_{p_{i+1}} \pi_1 \mathrm{ad}_u p_{i+2}) \\
&= \int K(\mathrm{ad}_{p_{i+1}} p_i, \pi_1 \mathrm{ad}_u p_{i+2}) \\
&= -\int K(\mathrm{ad}_{p_{i+1}} p_i, \pi_1 \mathrm{ad}_{p_{i+2}} u) \\
&= -\int K(\mathrm{ad}_{p_{i+1}} p_i, \mathrm{ad}_{p_{i+2}} u - \pi_0 \mathrm{ad}_{p_{i+2}} u) \\
&= \int K(\mathrm{ad}_{p_{i+2}} \mathrm{ad}_{p_{i+1}} p_i, u) + \int K(\mathrm{ad}_{p_{i+1}} p_i, \pi_0 \mathrm{ad}_{p_{i+2}} u) \\
&= \int K(\mathrm{ad}_{p_{i+1}} p_i, \pi_0 \mathrm{ad}_{p_{i+2}} u) \\
&= \int K(\pi_0 \mathrm{ad}_{p_{i+1}} p_i, \pi_0 \mathrm{ad}_{p_{i+2}} u) \\
&= \int K(\mathrm{ad}_{\rho_0 p_{i+1}} \rho_0 p_i, \mathrm{ad}_{\rho_0 p_{i+2}} \rho_0 u) \\
&= -\int K(\mathrm{ad}_{\rho_0 p_{i+2}} \mathrm{ad}_{\rho_0 p_{i+1}} \rho_0 p_i, \rho_0 u) \\
&= 0.
\end{aligned}$$

Notice that we have used the equality

$$\int K(\mathrm{ad}_{p_{i+2}} \mathrm{ad}_{p_{i+1}} p_i, u) = 0,$$

using the Jacobi identity and also the fact that

$$\pi_0 \mathrm{ad}_{p_{i+1}} p_i = \mathrm{ad}_{\rho_0 p_{i+1}} \rho_0 p_i. \tag{6.1.7}$$

This concludes the proof of the lemma. ■

Lemma 6.1.5. $[H_1, H_2] = 0$

Proof.

$$
\begin{aligned}
[H_1, H_2](p_1, p_2, p_3) &= -\int K(p_i, \pi_1 \mathrm{ad}_{p_{i+2,x}} p_{i+1}) \\
&= -\int K(p_i, \mathrm{ad}_{p_{i+2,x}} p_{i+1}) \\
&= -\frac{1}{3} \int D_x K(p_i, \mathrm{ad}_{p_{i+2}} p_{i+2}) \\
&= 0.
\end{aligned}
$$

This concludes the proof of the lemma. ■

Lemma 6.1.6. $[H_1, H_3] = 0$

Proof.

$$
\begin{aligned}
&[H_1, H_3](p_1, p_2, p_3) = \\
&= -\int K(p_i, \mathrm{ad}_{p_{i+2,x}} D_x^{-1} \pi_0 \mathrm{ad}_u p_{i+1}) - \int K(p_i, \mathrm{ad}_u D_x^{-1} \pi_0 \mathrm{ad}_{p_{i+2,x}} p_{i+1}) \\
&= \int K(\mathrm{ad}_{p_{i+2,x}} p_i, D_x^{-1} \pi_0 \mathrm{ad}_u p_{i+1}) + \int K(\mathrm{ad}_u p_i, D_x^{-1} \pi_0 \mathrm{ad}_{p_{i+2,x}} p_{i+1}) \\
&= -\int K(D_x^{-1} \mathrm{ad}_{p_{i+2,x}} p_i, \pi_0 \mathrm{ad}_u p_{i+1}) + \int K(\pi_0 \mathrm{ad}_u p_{i+1}, D_x^{-1} \mathrm{ad}_{p_{i,x}} p_{i+2}) \\
&= \int K(\pi_0 \mathrm{ad}_u p_{i+1}, D_x^{-1} \mathrm{ad}_{p_i} p_{i+2,x}) + \int K(\pi_0 \mathrm{ad}_u p_{i+1}, D^{-1} \mathrm{ad}_{p_{i,x}} p_{i+2,x}) \\
&= \int K(\pi_0 \mathrm{ad}_u p_{i+1}, \mathrm{ad}_{p_i} p_{i+2}) \\
&= \int K(\pi_0 \mathrm{ad}_u p_{i+1}, \pi_0 \mathrm{ad}_{p_i} p_{i+2}) \\
&= \int K(\mathrm{ad}_{\rho_0 u} \rho_0 p_{i+1}, \mathrm{ad}_{\rho_0 p_i} \rho_0 p_{i+2}) \\
&= \int K(\rho_0 u, \mathrm{ad}_{\rho_0 p_{i+1}} \mathrm{ad}_{\rho_0 p_i} \rho_0 p_{i+2}) \\
&= 0.
\end{aligned}
$$

Here again (6.1.7) plays a key role. ■

Now we obtain the main result out of these lemmas.

Theorem 6.1.7. The operator

$$H = D_x - \pi_1 \mathrm{ad}_u - \mathrm{ad}_u D^{-1} \pi_0 \mathrm{ad}_u$$

is a Hamiltonian operator.

Now we can simply prove that the operator $\mathfrak{H}\mathfrak{A}^*$ or its systematic form in terms of the Lie bracket and projection as $\hat{\mathfrak{H}}\hat{\mathfrak{A}}^*$ is also Hamiltonian. In fact we have that

$$\begin{aligned}
& \hat{\mathfrak{H}}\hat{\mathfrak{A}}^* = (D_x - \pi_1\mathrm{ad}_u - \mathrm{ad}_u D_x^{-1}\pi_0\mathrm{ad}_u)(\rho_0 + 2\rho_1 - \rho_1 D_x^{-1}\pi_1\mathrm{ad}_u) \\
= \;& (D_x - \pi_1\mathrm{ad}_u - \mathrm{ad}_u D_x^{-1}\pi_0\mathrm{ad}_u)(\rho_0 + 2\rho_1) - \rho_1\pi_1\mathrm{ad}_u + \pi_1\mathrm{ad}_u\rho_1 D_x^{-1}\pi_1\mathrm{ad}_u \\
= \;& (D_x - \pi_1\mathrm{ad}_u - \mathrm{ad}_u D_x^{-1}\pi_0\mathrm{ad}_u)(\rho_0 + 2\rho_1) - \rho_1\pi_1\mathrm{ad}_u + \mathrm{ad}_u\rho_1 D_x^{-1}\pi_1\mathrm{ad}_u \\
= \;& (D_x - \pi_1\mathrm{ad}_u - \mathrm{ad}_u D_x^{-1}\pi_0\mathrm{ad}_u)(\rho_0 + 2\rho_1) - \rho_1\pi_1\mathrm{ad}_u + \mathrm{ad}_u D_x^{-1}\rho_1\pi_1\mathrm{ad}_u \\
= \;& D_x - \pi_1\mathrm{ad}_u - \mathrm{ad}_u D_x^{-1}\pi_0\mathrm{ad}_u + \\
& D_x\rho_1 - \pi_1\mathrm{ad}_u\rho_1 + \rho_1\pi_1\mathrm{ad}_u + \mathrm{ad}_u D_x^{-1}\rho_1\pi_1\mathrm{ad}_u.
\end{aligned}$$

Let us write $\hat{\mathfrak{H}}\hat{\mathfrak{A}}^* = C_1 + C_2$ in which

$$C_1 = D_x - \pi_1\mathrm{ad}_u - \mathrm{ad}_u D_x^{-1}\pi_0\mathrm{ad}_u,$$

and

$$C_2 = D_x\rho_1 - \pi_1\mathrm{ad}_u\rho_1 + \rho_1\pi_1\mathrm{ad}_u + \mathrm{ad}_u D_x^{-1}\rho_1\pi_1\mathrm{ad}_u.$$

We already proved that C_1 is Hamiltonian, that is, $[C_1, C_1] = 0$. Similarly we can prove that $[C_2, C_2] = 0$. It is not difficult to show that $[C_1, C_2] = 0$. Hence we can simply conclude the following lemma.

Theorem 6.1.8. $\hat{\mathfrak{H}}\hat{\mathfrak{A}}^*$ is also Hamiltonian.

Remark 6.1.9. Indeed this representation of the Hamiltonian operator is a specific case of typical Hamiltonian operators, see for instance a series of papers [69], [70] and [68] and [13].

6.2 Symplectic operator

As announced in Chapter 4, this section is devoted to the proof of the fact that the operator $\mathfrak{A}^{-1*}\mathfrak{I}\mathfrak{A}^{-1}$ is symplectic. In order to do so, is enough to show that the Lie algebra form $\hat{\mathfrak{A}}^{-1*}\hat{\mathfrak{I}}\hat{\mathfrak{A}}^{-1}$ of this operator is symplectic. This will be done in a few steps. First we give some identities which we use later on.

Proposition 6.2.1. Using the notation of Section 4.4, we have that

1. $(\frac{1}{2}\rho_1 + \rho_0)\pi_1\mathrm{ad}_a^2 q = -q.$
2. $\pi_1\mathrm{ad}_a\mathrm{ad}_u\mathrm{ad}_a\pi_1 q = -\mathrm{ad}_u\rho_1 q - \pi_1\mathrm{ad}_{\rho_0 u}\rho_0 q,$

where the matrix q and u have the form

$$q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & q & -\overline{\mathbf{q}}^t \\ 0 & \mathbf{q} & 0 \end{pmatrix}, \quad u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & u & -\overline{\mathbf{u}}^t \\ 0 & \mathbf{u} & 0 \end{pmatrix}. \tag{6.2.1}$$

We have removed here the hat sign on top of the symbols for the matrices to have a simpler looking notation.

Now using this proposition, the operator $\hat{\mathfrak{I}}$ can be expressed and simplified as in the following lemma.

Lemma 6.2.2. The operator $\hat{\mathfrak{I}}$ can be decomposed as

$$\hat{\mathfrak{I}} = S_0 + S_1 + S_2,$$

in which

$$S_0 = -\frac{1}{2}uD_x^{-1}K(u,\cdot), \quad S_1 = \frac{1}{2}D_x\rho_1 - \frac{1}{4}\mathrm{ad}_{\rho_1 u}\rho_1,$$

and

$$S_2 = D_x\rho_0 - \frac{1}{2}\mathrm{ad}_{\rho_0 u}\rho_1 - \frac{1}{2}\pi_1\mathrm{ad}_{\rho_0 u}\rho_0.$$

Proof.

$$\begin{aligned}
&\hat{\mathfrak{I}} \\
=\ & -\frac{1}{2}uD_x^{-1}K(u,\cdot) - (\frac{1}{2}\rho_1 + \rho_0)\pi_1\mathrm{ad}_a(D_x - \mathrm{ad}_u)\mathrm{ad}_a\pi_1(\frac{1}{2}\rho_1 + \rho_0) \\
=\ & -\frac{1}{2}uD_x^{-1}K(u,\cdot) - (\frac{1}{2}\rho_1 + \rho_0)\pi_1\mathrm{ad}_a^2\pi_1(\frac{1}{2}\rho_1 + \rho_0)D_x \\
& +(\frac{1}{2}\rho_1 + \rho_0)\pi_1\mathrm{ad}_a\mathrm{ad}_u\mathrm{ad}_a\pi_1(\frac{1}{2}\rho_1 + \rho_0) \\
=\ & -\frac{1}{2}uD_x^{-1}K(u,\cdot) + (\frac{1}{2}\rho_1 + \rho_0)D_x \\
& +(\frac{1}{2}\rho_1 + \rho_0)(-\mathrm{ad}_u\frac{1}{2}\rho_1 - \pi_1\mathrm{ad}_{\rho_0 u}\rho_0) \\
=\ & -\frac{1}{2}uD_x^{-1}K(u,\cdot) + (\frac{1}{2}\rho_1 + \rho_0)D_x \\
& -\frac{1}{4}\mathrm{ad}_{\rho_1 u}\rho_1 - \frac{1}{2}\mathrm{ad}_{\rho_0 u}\rho_1 - \frac{1}{2}\pi_1\mathrm{ad}_{\rho_0 u}\rho_0
\end{aligned}$$

■

In the following proposition we compute $\hat{\mathfrak{A}}^{-1}, \hat{\mathfrak{A}}^{-1*}$ and also prove that the operator $\hat{\mathfrak{I}}$ and consequently the operator $\hat{\mathfrak{A}}^{-1*}\hat{\mathfrak{I}}\hat{\mathfrak{A}}^{-1}$ is anti-symmetric.

Proposition 6.2.3. We do have that

1. $\hat{\mathfrak{A}}^{-1} = D_xB\rho_1 + (\mathrm{ad}_{\rho_0 u}B\rho_1 + \rho_0)$, in which $B = (2D_x - \mathrm{ad}_{\rho_1 u})^{-1}$.
2. $B^* = -B$.
3. $\hat{\mathfrak{A}}^{-1*} = BD_x\rho_1 + B\pi_1\mathrm{ad}_{\rho_0 u}\rho_0 + \rho_0$.
4. $S_i^* = -S_i$ for $i = 0, 1, 2$.

Proof. 1. We see that

$$\begin{aligned}
\hat{\mathfrak{A}}\hat{\mathfrak{A}}^{-1} &= (\rho_0 + 2\rho_1 - \mathrm{ad}_u D_x^{-1}\rho_1)(D_x B\rho_1 + (\mathrm{ad}_{\rho_0} B\rho_1 + \rho_0)) \\
&= 2D_x B\rho_1 + \mathrm{ad}_{\rho_0 u} B\rho_1 + \rho_0 - \mathrm{ad}_u B\rho_1 \\
&= (2D_x - \mathrm{ad}_u + \mathrm{ad}_{\rho_0 u}) B\rho_1 + \rho_0 \\
&= (2D_x - \mathrm{ad}_{\rho_1 u}) B\rho_1 + \rho_0 \\
&= \rho_1 + \rho_0.
\end{aligned}$$

Similarly we have that

$$\begin{aligned}
\hat{\mathfrak{A}}^{-1}\hat{\mathfrak{A}} &= (D_x B\rho_1 + (\mathrm{ad}_{\rho_0} B\rho_1 + \rho_0))(\rho_0 + 2\rho_1 - \mathrm{ad}_u D_x^{-1}\rho_1) \\
&= 2D_x B\rho_1 - D_x B \mathrm{ad}_{\rho_1 u} D_x^{-1}\rho_1 \\
&\quad + 2\mathrm{ad}_{\rho_0 u} B\rho_1 - \mathrm{ad}_{\rho_0 u} B \mathrm{ad}_{\rho_1 u} D_x^{-1}\rho_1 + \rho_0 - \mathrm{ad}_{\rho_0 u} D_x^{-1} \\
&= D_x B(2D_x - \mathrm{ad}_{\rho_1 u}) D_x^{-1}\rho_1 \\
&\quad + \mathrm{ad}_{\rho_0 u} B(2D_x - \mathrm{ad}_{\rho_1 u}) D_x^{-1}\rho_1 + \rho_0 - \mathrm{ad}_{\rho_0 u} D_x^{-1} \\
&= \rho_1 + \rho_0.
\end{aligned}$$

2. We can write the operator B as follows.

$$\begin{aligned}
B &= \frac{1}{2}(I - \frac{1}{2}D_x^{-1}\mathrm{ad}_{\rho_1 u})^{-1} D_x^{-1} \\
&= \frac{1}{2}(I + \frac{1}{2}D_x^{-1}\mathrm{ad}_{\rho_1 u} + \frac{1}{4}D_x^{-1}\mathrm{ad}_{\rho_1 u} D_x^{-1}\mathrm{ad}_{\rho_1 u} + \ldots) D_x^{-1} \\
&= \frac{1}{2}(D_x^{-1} + \frac{1}{2}D_x^{-1}\mathrm{ad}_{\rho_1 u} D_x^{-1} + \frac{1}{4}D_x^{-1}\mathrm{ad}_{\rho_1 u} D_x^{-1}\mathrm{ad}_{\rho_1 u} D_x^{-1} + \ldots).
\end{aligned}$$

Notice that the summation is finite when applied to elements of the Lie algebra $\mathfrak{E}$. Now using the integration by parts and (6.1.4), we find that

$$\begin{aligned}
\int K(Bp, q) &= \int K(\frac{1}{2}(D_x^{-1} + \frac{1}{2}D_x^{-1}\mathrm{ad}_{\rho_1 u} D_x^{-1} \\
&\quad + \frac{1}{4}D_x^{-1}\mathrm{ad}_{\rho_1 u} D_x^{-1}\mathrm{ad}_{\rho_1 u} D_x^{-1} + \ldots)p, q) \\
&= \int K(p, -\frac{1}{2}(D_x^{-1} + \frac{1}{2}D_x^{-1}\mathrm{ad}_{\rho_1 u} D_x^{-1} \\
&\quad + \frac{1}{4}D_x^{-1}\mathrm{ad}_{\rho_1 u} D_x^{-1}\mathrm{ad}_{\rho_1 u} D_x^{-1} + \ldots)q) \\
&= \int K(p, -Bq).
\end{aligned}$$

Hence we proved that $B^* = -B$.

3. We can compute the adjoint of the operator $\hat{\mathfrak{A}}^{-1}$ as follows.

$$
\begin{aligned}
\int K(\hat{\mathfrak{A}}^{-1}p,q) &= \int K((D_xB\rho_1+\mathrm{ad}_{\rho_0 u}B\rho_1+\rho_0)p,q)\\
&= -\int K(Bp,D_x\rho_1 q)-K(B\rho_1 p,\pi_1\mathrm{ad}_{\rho_0 u}\rho_0 q)+K(p,\rho_0 q)\\
&= +\int K(p,BD_x\rho_1 q)+K(p,B\pi_1\mathrm{ad}_{\rho_0 u}\rho_0 q)+K(p,\rho_0 q)\\
&= +\int K(p,(BD_x\rho_1+B\pi_1\mathrm{ad}_{\rho_0 u}\rho_0+\rho_0)q).
\end{aligned}
$$

Hence by definition $\hat{\mathfrak{A}}^{-1*}=BD_x\rho_1+B\pi_1\mathrm{ad}_{\rho_0 u}\rho_0+\rho_0$.

4. For this equality, we use Lemma 1.4.2. See Remark 6.2.4 below.

$$
\begin{aligned}
\int K(S_0p,q) &= \int K\Big(-\frac{1}{2}uD_x^{-1}K(u,p),q\Big)\\
&= -\int \frac{1}{2}K(u,q).D_x^{-1}K(u,p)\\
&= +\int \frac{1}{2}D_x^{-1}K(u,q).K(u,p)\\
&= +\int \frac{1}{2}K(uD_x^{-1}K(u,q),p)\\
&= \int K\Big(-(\frac{1}{2}uD_x^{-1}K(u,q)),p\Big)
\end{aligned}
$$

Hence by the definition of adjoint operator, we see that $S_0^*=-S_0$. Notice that this can be considered as an example of how we work with such expressions. For instance the technique used here can be seen in the lemmas afterwards proving that $\hat{\mathfrak{A}}^{-1*}\hat{\mathfrak{I}}\hat{\mathfrak{A}}^{-1}$ is symplectic, or applied to the current case, more precisely $\hat{\mathfrak{A}}^{-1*}S_0\hat{\mathfrak{A}}^{-1}$ is symplectic.

Now we compute the adjoint operator of S_1 :

$$
\begin{aligned}
\int K(S_1p,q) &= \int K\Big((\frac{1}{2}D_x\rho_1-\frac{1}{4}\mathrm{ad}_{\rho_1 u}\rho_1)p,q\Big)\\
&= +\int K(\frac{1}{2}D_x\rho_1 p,\rho_1 q)-\frac{1}{4}K(\mathrm{ad}_{\rho_1 u}\rho_1 p,\rho_1 q)\\
&= -\int K(\rho_1 p,\frac{1}{2}D_x\rho_1 q)+K(\rho_1 p,\frac{1}{4}\mathrm{ad}_{\rho_1 u}\rho_1 q)\\
&= -\int K(p,\frac{1}{2}D_x\rho_1 q)+K(p,\frac{1}{4}\mathrm{ad}_{\rho_1 u}\rho_1 q)\\
&= \int K\Big(p,-(\frac{1}{2}D_x\rho_1-\frac{1}{4}\mathrm{ad}_{\rho_1 u}\rho_1)q\Big)
\end{aligned}
$$

So that $S_1^*=-S_1$.

Similarly we can manipulate S_2^* as follows using the same technics.

$$\begin{aligned}
& \int K(S_2 p, q) \\
= & \int K\Big((D_x\rho_0 - \frac{1}{2}\mathrm{ad}_{\rho_0 u}\rho_1 - \frac{1}{2}\pi_1 \mathrm{ad}_{\rho_0 u}\rho_0)p, q\Big) \\
= & \int K(D_x\rho_0 p, \rho_0 q) - \frac{1}{2}K(\mathrm{ad}_{\rho_0 u}\rho_1 p, \rho_0 q) - \frac{1}{2}K(\pi_1 \mathrm{ad}_{\rho_0 u}\rho_0 p, \rho_1 q) \\
= & -\int K(\rho_0 p, D_x\rho_0 q) + \frac{1}{2}K(\rho_1 p, \pi_1 \mathrm{ad}_{\rho_0 u}\rho_0 q) + \frac{1}{2}K(\rho_0 p, \mathrm{ad}_{\rho_0 u}\rho_1 q) \\
= & -\int K(p, D_x\rho_0 q) + \frac{1}{2}K(p, \pi_1 \mathrm{ad}_{\rho_0 u}\rho_0 q) + \frac{1}{2}K(p, \mathrm{ad}_{\rho_0 u}\rho_1 q) \\
= & \int K\Big(p, (-D_x\rho_0 + \frac{1}{2}\pi_1 \mathrm{ad}_{\rho_0 u}\rho_0 + \frac{1}{2}\mathrm{ad}_{\rho_0 u}\rho_1)q\Big).
\end{aligned}$$

Thus $S_2^* = -D_x\rho_0 + \frac{1}{2}\pi_1 \mathrm{ad}_{\rho_0 u}\rho_0 + \frac{1}{2}\mathrm{ad}_{\rho_0 u}\rho_1 = -S_2$.

∎

Remark 6.2.4. There is a technical point here: using Lemma 1.4.2, if we have ρ_i in the first component of the Killing form, then that would move to the other component. For example $K(\rho_0 p, q) = K(p, \rho_0 q)$ or $K(\rho_1 p, \mathrm{ad}_{\rho_0 u}\rho_0 q) = K(\rho_1 p, \pi_1 \mathrm{ad}_{\rho_0 u}\rho_0 q)$ and so on.

The next lemma is essential in what follows, in particular in proving that the operator mentioned above is symplectic.

Lemma 6.2.5. The following identity holds for the operator B and arbitrary matrices p, q, r of the form (6.2.1):

$$\int K(\mathrm{ad}_{B\rho_1 p} B\rho_1 r, \rho_1 q) = -\int K(\mathrm{ad}_{\rho_1 p} B\rho_1 r - \mathrm{ad}_{\rho_1 r} B\rho_1 p, B\rho_1 q).$$

Proof. By definition and the Leibniz rule we have that

$$\begin{aligned}
& 2D_x K(\mathrm{ad}_{B\rho_1 p} B\rho_1 r, B\rho_1 q) \\
= & K(\mathrm{ad}_{2D_x B\rho_1 p} B\rho_1 r, B\rho_1 q) \\
+ & K(\mathrm{ad}_{B\rho_1 p} 2D_x B\rho_1 r, B\rho_1 q) \\
+ & K(\mathrm{ad}_{B\rho_1 p} B\rho_1 r, 2D_x B\rho_1 q) \\
= & K(\mathrm{ad}_{\rho_1 p} B\rho_1 r, B\rho_1 q) + K(\mathrm{ad}_{\mathrm{ad}_{\rho_1 u} B\rho_1 p} B\rho_1 r, B\rho_1 q) \\
+ & K(\mathrm{ad}_{B\rho_1 p}\rho_1 r, B\rho_1 q) + K(\mathrm{ad}_{B\rho_1 p}\mathrm{ad}_{\rho_1 u} B\rho_1 r, B\rho_1 q) \\
+ & K(\mathrm{ad}_{B\rho_1 p} B\rho_1 r, \rho_1 q) + K(\mathrm{ad}_{B\rho_1 p} B\rho_1 r, \mathrm{ad}_{\rho_1 u} B\rho_1 q) \\
= & K(\mathrm{ad}_{\rho_1 p} B\rho_1 r, B\rho_1 q) - K(\mathrm{ad}_{B\rho_1 r}\mathrm{ad}_{\rho_1 u} B\rho_1 p, B\rho_1 q) \\
+ & K(\mathrm{ad}_{B\rho_1 p}\rho_1 r, B\rho_1 q) + K(\mathrm{ad}_{B\rho_1 p}\mathrm{ad}_{\rho_1 u} B\rho_1 r, B\rho_1 q) \\
+ & K(\mathrm{ad}_{B\rho_1 p} B\rho_1 r, \rho_1 q) - K(\mathrm{ad}_{\rho_1 u}\mathrm{ad}_{B\rho_1 p} B\rho_1 r, B\rho_1 q).
\end{aligned}$$

We have used the fact that $2D_x B\rho_1 p = \rho_1 p + \mathrm{ad}_{\rho_1 u} B\rho_1 p$.

Now using the Jacobi identity we obtain

$$\begin{aligned} 2D_xK(\mathrm{ad}_{B\rho_1p}B\rho_1r, B\rho_1q) &= K(\mathrm{ad}_{\rho_1p}B\rho_1r, B\rho_1q) + K(\mathrm{ad}_{B\rho_1p}\rho_1r, B\rho_1q) \\ &+ K(\mathrm{ad}_{B\rho_1p}B\rho_1r, \rho_1q). \end{aligned}$$

Hence by definition

$$\int K(\mathrm{ad}_{\rho_1p}B\rho_1r, B\rho_1q) + K(\mathrm{ad}_{B\rho_1p}\rho_1r, B\rho_1q) + K(\mathrm{ad}_{B\rho_1p}B\rho_1r, \rho_1q) = 0.$$

The statement is proved. ■

Now we yield the operators $\hat{\mathfrak{A}}^{-1*}S_i\hat{\mathfrak{A}}^{-1}$ explicitly.

Lemma 6.2.6. 1.

$$\mathfrak{A}^{-1*}S_2\mathfrak{A}^{-1} = \frac{1}{2}B\pi_1\mathrm{ad}_{\mathrm{ad}_{\rho_0u}D_x\rho_0u}B\rho_1 - \frac{1}{2}BD_x\pi_1\mathrm{ad}_{\rho_0u}\rho_0$$
$$+B\pi_1\mathrm{ad}_{\rho_0u}D_x\rho_0 + D_x\mathrm{ad}_{\rho_0u}B\rho_1 + D_x\rho_0 - \frac{1}{2}\mathrm{ad}_{\rho_0u}D_xB\rho_1.$$

2. $\mathfrak{A}^{-1*}S_1\mathfrak{A}^{-1} = \frac{1}{2}BD_x^3B\rho_1 - \dfrac{1}{4}BD_x\mathrm{ad}_{\rho_1u}D_xB\rho_1.$

3. For the operator S_0 we have that

$$\begin{aligned} \mathfrak{A}^{-1*}S_0\mathfrak{A}^{-1} &= -\frac{1}{2}BD_x\Big((\rho_1u)D_x^{-1}K(u, D_xB\rho_1 + \rho_0)\Big) \\ &-\frac{1}{2}(\rho_0u)D_x^{-1}K(u, D_xB\rho_1 + \rho_0). \end{aligned}$$

Proof. 1. We have that

$$\begin{aligned} &\mathfrak{A}^{-1*}S_2\mathfrak{A}^{-1} \\ =\ & -\frac{1}{2}BD_x\pi_1\mathrm{ad}_{\rho_0u}\mathrm{ad}_{\rho_0u}B\rho_1 - \frac{1}{2}BD_x\pi_1\mathrm{ad}_{\rho_0u}\rho_0 \\ +\ & B\pi_1\mathrm{ad}_{\rho_0u}D_x\mathrm{ad}_{\rho_0u}B\rho_1 + B\pi_1\mathrm{ad}_{\rho_0u}D_x\rho_0 \\ -\ & \frac{1}{2}B\pi_1\mathrm{ad}_{\rho_0u}\mathrm{ad}_{\rho_0u}D_xB\rho_1 \\ +\ & D_x\mathrm{ad}_{\rho_0u}B\rho_1 + D_x\rho_0 - \frac{1}{2}\mathrm{ad}_{\rho_0u}D_xB\rho_1 \\ =\ & -\frac{1}{2}B\pi_1\mathrm{ad}_{D_x\rho_0u}\mathrm{ad}_{\rho_0u}B\rho_1 + \frac{1}{2}BD_x\pi_1\mathrm{ad}_{\rho_0u}\mathrm{ad}_{D_x\rho_0u}B\rho_1 \\ -\ & \frac{1}{2}BD_x\pi_1\mathrm{ad}_{\rho_0u}\rho_0 \\ +\ & B\pi_1\mathrm{ad}_{\rho_0u}D_x\rho_0 + D_x\mathrm{ad}_{\rho_0u}B\rho_1 + D_x\rho_0 - \frac{1}{2}\mathrm{ad}_{\rho_0u}D_xB\rho_1 \\ =\ & \frac{1}{2}B\pi_1\mathrm{ad}_{\mathrm{ad}_{\rho_0u}D_x\rho_0u}B\rho_1 - \frac{1}{2}BD_x\pi_1\mathrm{ad}_{\rho_0u}\rho_0 \\ +\ & B\pi_1\mathrm{ad}_{\rho_0u}D_x\rho_0 + D_x\mathrm{ad}_{\rho_0u}B\rho_1 + D_x\rho_0 - \frac{1}{2}\mathrm{ad}_{\rho_0u}D_xB\rho_1. \end{aligned}$$

We use the Jacobi identity in the last expression on the first two terms, so that

$$\begin{aligned} & -\frac{1}{2}B\pi_1 \mathrm{ad}_{D_x\rho_0 u}\mathrm{ad}_{\rho_0 u}B\rho_1 + \frac{1}{2}BD_x\pi_1\mathrm{ad}_{\rho_0 u}\mathrm{ad}_{D_x\rho_0 u}B\rho_1 \\ = \; & \frac{1}{2}B\pi_1\mathrm{ad}_{\mathrm{ad}_{\rho_0 u}D_x\rho_0 u}B\rho_1. \end{aligned}$$

2. The second identity is simple.

3. Now for the third one, notice that the image of $\mathrm{ad}_{\rho_0 u}B\rho_1$ is contained in the image of ρ_0, since the image of $B\rho_1$ is contained in the image of ρ_1. Hence we do have that for instance $K(u, \mathrm{ad}_{\rho_0 u}B\rho_1\cdot) = 0$ using the invariance property of the Killing form under adjoint action (6.1.4). Taking these facts into account we see that

$$\begin{aligned} & \mathfrak{A}^{-1*}S_0\mathfrak{A}^{-1} = \\ = \; & (BD_x\rho_1 + B\pi_1\mathrm{ad}_{\rho_0 u}\rho_0 + \rho_0)(-\frac{1}{2}uD_x^{-1}K(u,\cdot))(D_xB\rho_1 + \mathrm{ad}_{\rho_0 u}B\rho_1 + \rho_0) \\ = \; & (BD_x\rho_1 + B\pi_1\mathrm{ad}_{\rho_0 u}\rho_0 + \rho_0)(-\frac{1}{2}uD_x^{-1}K(u, D_xB\rho_1\cdot + \rho_0\cdot)) \\ = \; & -\frac{1}{2}BD_x\Big((\rho_1 u)D_x^{-1}K(u, D_xB\rho_1 + \rho_0)\Big) \\ & -\frac{1}{2}(\rho_0 u)D_x^{-1}K(u, D_xB\rho_1 + \rho_0). \end{aligned}$$

This concludes the proof of the three statements of the lemma. ■

As we discussed in Chapter 2, in order to prove that an operator is symplectic, we first need to compute its Fréchet derivative.

Lemma 6.2.7. The Fréchet derivatives of $\hat{\mathfrak{A}}^{-1*}S_0\hat{\mathfrak{A}}^{-1}, \hat{\mathfrak{A}}^{-1*}S_1\hat{\mathfrak{A}}^{-1}$ and $\hat{\mathfrak{A}}^{-1*}S_2\hat{\mathfrak{A}}^{-1}$ are expressed as follows:

1.

$$\begin{aligned} & D_{\hat{\mathfrak{A}}^{-1*}S_2\hat{\mathfrak{A}}^{-1}}[p_{i+2}] \\ = \; & +\frac{1}{2}B\mathrm{ad}_{\rho_1 p_{i+2}}B\pi_1\mathrm{ad}_{\mathrm{ad}_{\rho_0 u}D_x\rho_0 u}B\rho_1 + \frac{1}{2}B\pi_1\mathrm{ad}_{\mathrm{ad}_{\rho_0 p_{i+2}}D_x\rho_0 u}B\rho_1 \\ & +\frac{1}{2}B\pi_1\mathrm{ad}_{\mathrm{ad}_{\rho_0 u}D_x\rho_0 p_{i+2}}B\rho_1 + \frac{1}{2}B\pi_1\mathrm{ad}_{\mathrm{ad}_{\rho_0 u}D_x\rho_0 u}B\mathrm{ad}_{\rho_1 p_{i+2}}B\rho_1 \\ & -\frac{1}{2}B\mathrm{ad}_{\rho_1 p_{i+2}}BD_x\pi_1\mathrm{ad}_{\rho_0 u}\rho_0 - \frac{1}{2}BD_x\pi_1\mathrm{ad}_{\rho_0 p_{i+2}}\rho_0 \\ & +B\mathrm{ad}_{\rho_1 p_{i+2}}B\pi_1\mathrm{ad}_{\rho_0 u}D_x\rho_0 + B\pi_1\mathrm{ad}_{\rho_0 p_{i+2}}D_x\rho_0 \\ & +D_x\mathrm{ad}_{\rho_0 p_{i+2}}B\rho_1 + D_x\mathrm{ad}_{\rho_0 u}B\mathrm{ad}_{\rho_1 p_{i+2}}B\rho_1 \\ & -\frac{1}{2}\mathrm{ad}_{\rho_0 p_{i+2}}D_xB\rho_1 - \frac{1}{2}\mathrm{ad}_{\rho_0 u}D_xB\mathrm{ad}_{\rho_1 p_{i+2}}B\rho_1. \end{aligned}$$

2.

$$\begin{aligned}
& D_{\hat{\mathfrak{A}}^{-1*}S_2\hat{\mathfrak{A}}^{-1}}[p_{i+2}] \\
= \; & +\frac{1}{2}B\mathrm{ad}_{\rho_1 p_{i+2}}BD_x^3B\rho_1 + \frac{1}{2}BD_x^3B\mathrm{ad}_{\rho_1 p_{i+2}}B\rho_1 \\
& -\frac{1}{4}B\mathrm{ad}_{\rho_1 p_{i+2}}BD_x\mathrm{ad}_{\rho_1 u}D_xB\rho_1 - \frac{1}{4}BD_x\mathrm{ad}_{\rho_1 u}D_xB\mathrm{ad}_{\rho_1 p_{i+2}}B\rho_1 \\
& -\frac{1}{4}BD_x\mathrm{ad}_{\rho_1 p_{i+2}}D_xB\rho_1.
\end{aligned}$$

3.

$$\begin{aligned}
& D_{\mathfrak{A}^{-1*}S_0\mathfrak{A}^{-1}}[p_{i+2}] \\
= \; & -\frac{1}{2}B\mathrm{ad}_{\rho_1 p_{i+2}}BD_x\Big((\rho_1 u)D_x^{-1}K(u, D_xB\rho_1 + \rho_0)\Big) \\
& -\frac{1}{2}BD_x\Big((\rho_1 p_{i+2})D_x^{-1}K(u, D_xB\rho_1 + \rho_0)\Big) \\
& -\frac{1}{2}BD_x\Big((\rho_1 u)D_x^{-1}K(p_{i+2}, D_xB\rho_1 + \rho_0)\Big) \\
& -\frac{1}{2}BD_x\Big((\rho_1 u)D_x^{-1}K(u, D_xB\mathrm{ad}_{\rho_1 p_{i+2}}B\rho_1)\Big) \\
& -\frac{1}{2}(\rho_0 p_{i+2})D_x^{-1}K(u, D_xB\rho_1 + \rho_0) \\
& -\frac{1}{2}(\rho_0 u)D_x^{-1}K(p_{i+2}, D_xB\rho_1 + \rho_0) \\
& -\frac{1}{2}(\rho_0 u)D_x^{-1}K(u, D_xB\mathrm{ad}_{\rho_1 p_{i+2}}B\rho_1).
\end{aligned}$$

Proof. We apply the Leibniz rule many times. Notice that we do have

$$D_{\mathrm{ad}_{\rho_1 u}}[p_{i+2}] = \mathrm{ad}_{\rho_1 p_{i+2}},$$

and

$$D_B[p_{i+2}] = B\mathrm{ad}_{\rho_1 p_{i+2}}B.$$

■

To prove that $\hat{\mathfrak{A}}^{-1*}\hat{\mathfrak{I}}\hat{\mathfrak{A}}^{-1}$ is symplectic, it is enough to show that the operators $\hat{\mathfrak{A}}^{-1*}S_i\hat{\mathfrak{A}}^{-1}$ for $i = 0, 1, 2$ are symplectic. As we have seen these operator are anti-symmetric. We start with the operator involving S_0. In order to shorten the proof, let us introduce some notation. We denote $\rho_1 p_i = q_i$ and $\rho_0 p_i = \mathbf{q}_i$. Also $\rho_0 u = v$ and $\rho_0 u = \mathbf{v}$.

Theorem 6.2.8. The operator $\hat{\mathfrak{A}}^{-1*}S_0\hat{\mathfrak{A}}^{-1}$ is symplectic.

Proof. As we discussed in Chapter 4, we need to prove that

$$[\mathfrak{A}^{-1*}S_1\mathfrak{A}^{-1}, \mathfrak{A}^{-1*}S_1\mathfrak{A}^{-1}] = \int K(p_i, D_{\hat{\mathfrak{A}}^{-1*}S_0\hat{\mathfrak{A}}^{-1}}[p_{i+2}](p_{i+1})) = 0,$$

where the bracket is the Schouten bracket. But we have that

$$\begin{aligned}
&[\mathfrak{A}^{-1*}S_0\mathfrak{A}^{-1}, \mathfrak{A}^{-1*}S_0\mathfrak{A}^{-1}] = \int K(p_i, D_{\hat{\mathfrak{A}}^{-1*}S_0\hat{\mathfrak{A}}^{-1}}[p_{i+2}](p_{i+1}))\\
= \quad &-\int K\Big(q_i, \frac{1}{2}B\mathrm{ad}_{q_{i+2}}BD_x\Big((v)D_x^{-1}K(v, D_xBq_{i+1})\Big)\Big)\\
&-\int K\Big(q_i, \frac{1}{2}B\mathrm{ad}_{q_{i+2}}BD_x\Big((v)D_x^{-1}K(\mathbf{v}, \mathbf{q}_{i+1})\Big)\Big)\\
&-\int K\Big(q_i, \frac{1}{2}BD_x\Big((q_{i+2})D_x^{-1}K(v, D_xBq_{i+1})\Big)\Big)\\
&-\int K\Big(q_i, \frac{1}{2}BD_x\Big((q_{i+2})D_x^{-1}K(\mathbf{v}, \mathbf{q}_{i+1})\Big)\Big)\\
&-\int K\Big(q_i, \frac{1}{2}BD_x\Big((v)D_x^{-1}K(q_{i+2}, D_xBq_{i+1})\Big)\Big)\\
&-\int K\Big(q_i, \frac{1}{2}BD_x\Big((v)D_x^{-1}K(\mathbf{q}_{i+2}, \mathbf{q}_{i+1})\Big)\Big)\\
&-\int K\Big(q_i, \frac{1}{2}BD_x\Big((v)D_x^{-1}K(v, D_xB\mathrm{ad}_{q_{i+2}}Bq_{i+1})\Big)\Big)\\
&-\int K\Big(\mathbf{q}_i, \frac{1}{2}(\mathbf{q}_{i+2})D_x^{-1}K(v, D_xBq_{i+1})\Big)\\
&-\int K\Big(\mathbf{q}_i, \frac{1}{2}(\mathbf{q}_{i+2})D_x^{-1}K(\mathbf{v}, \mathbf{q}_{i+1})\Big)\\
&-\int K\Big(\mathbf{q}_i, \frac{1}{2}(\mathbf{v})D_x^{-1}K(q_{i+2}, D_xBq_{i+1})\Big)\\
&-\int K\Big(\mathbf{q}_i, \frac{1}{2}(\mathbf{v})D_x^{-1}K(\mathbf{q}_{i+2}, \mathbf{q}_{i+1})\Big)\\
&-\int K\Big(\mathbf{q}_i, \frac{1}{2}(\mathbf{v})D_x^{-1}K(v, D_xB\mathrm{ad}_{q_{i+2}}Bq_{i+1})\Big). \qquad (6.2.2)
\end{aligned}$$

We now choose subexpressions and show they are zero.

1. The first expression to be considered is

$$\int K\Big(\mathbf{q}_i, -\frac{1}{2}(\mathbf{q}_{i+2})D_x^{-1}K(\mathbf{v}, \mathbf{q}_{i+1})\Big) + \int K\Big(\mathbf{q}_i, -\frac{1}{2}(\mathbf{v})D_x^{-1}K(\mathbf{q}_{i+2}, \mathbf{q}_{i+1})\Big).$$

The first and second terms of this expression become, respectively,

$$-\int \frac{1}{2}K(\mathbf{q}_i, \mathbf{q}_{i+2}).D_x^{-1}K(\mathbf{v}, \mathbf{q}_{i+1}), \quad -\int \frac{1}{2}K(\mathbf{q}_i, \mathbf{v}).D_x^{-1}K(\mathbf{q}_{i+2}, \mathbf{q}_{i+1}).$$

That is because a real valued expression such as $D_x^{-1}K(\mathbf{v},\mathbf{q}_{i+1})$ can pulled out of the integrand

$$-K\Big(\mathbf{q}_i,\frac{1}{2}(\mathbf{q}_{i+2})D_x^{-1}K(\mathbf{v},\mathbf{q}_{i+1})\Big).$$

Then we apply integration by part and the shifting rule. We obtain the following terms:

$$+\int\frac{1}{2}D_x^{-1}K(\mathbf{q}_i,\mathbf{q}_{i+2}).K(\mathbf{v},\mathbf{q}_{i+1}),\quad -\int\frac{1}{2}K(\mathbf{q}_{i+1},\mathbf{v}).D_x^{-1}K(\mathbf{q}_i,\mathbf{q}_{i+2}).$$

Now it is clear that the expression we chose vanishes.

2. Similar to the first one, we see that the expression

$$\begin{aligned}&\int K\Big(\mathbf{q}_i,-\frac{1}{2}(\mathbf{q}_{i+2})D_x^{-1}K(v,D_xBq_{i+1})\Big)\\&+\int K\Big(q_i,-\frac{1}{2}BD_x\Big((v)D_x^{-1}K(\mathbf{q}_{i+2},\mathbf{q}_{i+1})\Big)\Big)\end{aligned}$$

will vanish as well:

$$\begin{aligned}&+\int K\Big(\mathbf{q}_i,-\frac{1}{2}(\mathbf{q}_{i+2})D_x^{-1}K(v,D_xBq_{i+1})\Big)\\&+\int K\Big(q_i,-\frac{1}{2}BD_x\Big((v)D_x^{-1}K(\mathbf{q}_{i+2},\mathbf{q}_{i+1})\Big)\Big)\\=&-\int\frac{1}{2}K(\mathbf{q}_i,\mathbf{q}_{i+2}).D_x^{-1}K(v,D_xBq_{i+1})\\&-\int\frac{1}{2}K(D_xBq_i,v).D_x^{-1}K(\mathbf{q}_{i+2},\mathbf{q}_{i+1})\\=&+\int\frac{1}{2}D_x^{-1}K(\mathbf{q}_i,\mathbf{q}_{i+2}).K(v,D_xBq_{i+1})\\&-\int\frac{1}{2}K(D_xBq_{i+1},v).D_x^{-1}K(\mathbf{q}_i,\mathbf{q}_{i+2})\\=&\ 0\end{aligned}$$

3. The next expression is:

$$\begin{aligned}&+\int K\Big(\mathbf{q}_i,-\frac{1}{2}(\mathbf{v})D_x^{-1}K(q_{i+2},D_xBq_{i+1})\Big)\\&+\int K\Big(q_i,-\frac{1}{2}BD_x\Big((q_{i+2})D_x^{-1}K(\mathbf{v},\mathbf{q}_{i+1})\Big)\Big)\\&+\int K\Big(\mathbf{q}_i,-\frac{1}{2}(\mathbf{v})D_x^{-1}K(v,D_xB\mathrm{ad}_{q_{i+2}}Bq_{i+1})\Big)\\&+\int K\Big(q_i,-\frac{1}{2}B\mathrm{ad}_{q_{i+2}}BD_x\Big((v)D_x^{-1}K(\mathbf{v},\mathbf{q}_{i+1})\Big)\Big),\end{aligned}$$

which can be simplified as follows.

$$\frac{1}{4}\int D_x^{-1}K(\mathbf{q}_{i+1},\mathbf{v}).K(q_i,\mathrm{ad}_v Bq_{i+2}) - K(\mathrm{ad}_v Bq_i, q_{i+2}).D_x^{-1}K(\mathbf{v},\mathbf{q}_{i+1})$$
$$-\frac{1}{4}\int D_x^{-1}K\Big(\mathbf{q}_{i+1},\mathbf{v}).K(q_i,\mathrm{ad}_v Bq_{i+2}) + K(\mathrm{ad}_v Bq_i, q_{i+2}).D_x^{-1}K(\mathbf{v},\mathbf{q}_{i+1}).$$

Now it is clear that the expression is zero. See for more details Appendix B.

4. The last expression is:

$$+\int K\Big(q_i, -\frac{1}{2}B\mathrm{ad}_{q_{i+2}}BD_x\Big((v)D_x^{-1}K(v, D_xBq_{i+1})\Big)\Big)$$
$$+\int K\Big(q_i, -\frac{1}{2}BD_x\Big((q_{i+2})D_x^{-1}K(v, D_xBq_{i+1})\Big)\Big)$$
$$+\int K\Big(q_i, -\frac{1}{2}BD_x\Big((v)D_x^{-1}K(q_{i+2}, D_xBq_{i+1})\Big)\Big)$$
$$+\int K\Big(q_i, -\frac{1}{2}BD_x\Big((v)D_x^{-1}K(v, D_xB\mathrm{ad}_{q_{i+2}}Bq_{i+1})\Big)\Big).$$

Again this expression is converted to the following one by using the same rule as in third item:

$$-\int \frac{1}{4}K(\mathrm{ad}_{q_{i+2}}Bq_i, v).D_x^{-1}K(v, D_xBq_{i+1})$$
$$+\int \frac{1}{4}K(\mathrm{ad}_{q_{i+2}}Bq_i, v).D_x^{-1}K(v, D_xBq_{i+1})$$
$$-\int \frac{1}{4}D_x^{-1}K(D_xBq_{i+1}, v).K(v, \mathrm{ad}_{q_i}Bq_{i+2})$$
$$+\int \frac{1}{4}D_x^{-1}K(D_xBq_{i+1}, v).K(v, \mathrm{ad}_{q_i}Bq_{i+2})$$

Now in the last expression every term is canceled by another, so that the whole expression vanishes. See for more detail on this computation Appendix B.

■

In what follows, we simply use the following identity as a rule.

$$2D_xB\rho_1 q = \rho_1 q + \mathrm{ad}_{\rho_1 u}B\rho_1 q. \qquad (6.2.3)$$

Theorem 6.2.9. The operator $\mathfrak{A}^{-1*}S_1\mathfrak{A}^{-1}$ is symplectic.

Proof. The Schouten bracket is computed and simplified, using (6.2.3), as follows.

$$
\begin{aligned}
&[\mathfrak{A}^{-1*}S_1\mathfrak{A}^{-1}, \mathfrak{A}^{-1*}S_1\mathfrak{A}^{-1}] = \int K(p_i, D_{\mathfrak{A}^{-1*}S_1\mathfrak{A}^{-1}}[p_{i+2}](p_{i+1}) \\
= &-\frac{1}{4}\int K\Big(q_i, (B\mathrm{ad}_{q_{i+2}}BD_x\mathrm{ad}_v D_x B + BD_x\mathrm{ad}_v D_x B\mathrm{ad}_{q_{i+2}}B)q_{i+1}\Big) \\
- &\frac{1}{4}\int K\Big(q_i, BD_x\mathrm{ad}_{q_{i+2}}D_x Bq_{i+1}\Big) \\
+ &\frac{1}{2}\int K\Big(q_i, B\mathrm{ad}_{q_{i+2}}BD_x^3Bq_{i+1} + BD_x^3B\mathrm{ad}_{q_{i+2}}Bq_{i+1}\Big) \\
= &\frac{1}{4}\int K\Big(B\mathrm{ad}_{q_{i+2}}Bq_i, D_x\mathrm{ad}_v D_x Bq_{i+1}\Big) - \frac{1}{4}\int K\Big(D_x\mathrm{ad}_v D_x Bq_i, B\mathrm{ad}_{q_{i+2}}Bq_{i+1}\Big) \\
- &\frac{1}{4}\int K\Big(D_x Bq_i, \mathrm{ad}_{q_{i+2}}D_x Bq_{i+1}\Big) \\
+ &\frac{1}{2}\int K\Big(D_x B\mathrm{ad}_{q_{i+2}}Bq_i, D_x^2Bq_{i+1}\Big) - \frac{1}{2}\int K\Big(D_x^2Bq_i, D_x B\mathrm{ad}_{q_{i+2}}Bq_{i+1}\Big) \\
= &\frac{1}{4}\int K\Big(B\mathrm{ad}_{q_{i+2}}Bq_i - B\mathrm{ad}_{q_i}Bq_{i+2}, D_x\mathrm{ad}_v D_x Bq_{i+1}\Big) \\
- &\frac{1}{16}\int K\Big(\mathrm{ad}_v Bq_{i+1} + q_{i+1}, \mathrm{ad}_{q_i}(\mathrm{ad}_v Bq_{i+2} + q_{i+2})\Big) \\
+ &\frac{1}{2}\int K\Big(D_x B\mathrm{ad}_{q_{i+2}}Bq_i - D_x B\mathrm{ad}_{q_i}Bq_{i+2}, D_x^2Bq_{i+1}\Big) \\
= &-\frac{1}{4}\int K\Big(D_x B\mathrm{ad}_{q_{i+2}}Bq_i - D_x B\mathrm{ad}_{q_i}Bq_{i+2}, \mathrm{ad}_v D_x Bq_{i+1}\Big) \\
- &\frac{1}{16}\int K\Big(\mathrm{ad}_v Bq_{i+1} + q_{i+1}, \mathrm{ad}_{q_i}q_{i+2}\Big) \\
- &\frac{1}{16}\int K\Big(q_{i+1}, \mathrm{ad}_{q_i}\mathrm{ad}_v Bq_{i+2}\Big) \\
- &\frac{1}{16}\int K\Big(\mathrm{ad}_v Bq_{i+1}, \mathrm{ad}_{q_i}\mathrm{ad}_v Bq_{i+2}\Big) \\
+ &\frac{1}{4}\int K\Big(D_x B\mathrm{ad}_{q_{i+2}}Bq_i - D_x B\mathrm{ad}_{q_i}Bq_{i+2}, D_x\mathrm{ad}_v Bq_{i+1} + D_x q_{i+1}\Big) \\
= &-\frac{1}{16}\int K\Big(\mathrm{ad}_v Bq_{i+1} + q_{i+1}, \mathrm{ad}_{q_i}q_{i+2}\Big) \\
- &\frac{1}{16}\int K\Big(q_{i+1}, \mathrm{ad}_{q_i}\mathrm{ad}_v Bq_{i+2}\Big) \\
- &\frac{1}{16}\int K\Big(\mathrm{ad}_v Bq_{i+1}, \mathrm{ad}_{q_i}\mathrm{ad}_v Bq_{i+2}\Big) \\
+ &\frac{1}{4}\int K\Big(D_x B\mathrm{ad}_{q_{i+2}}Bq_i - D_x B\mathrm{ad}_{q_i}Bq_{i+2}, \mathrm{ad}_{D_x v}Bq_{i+1} + D_x q_{i+1}\Big)
\end{aligned}
$$

In Appendix B we have simplified the terms in the last expression that has been found. There, in the Lemmas B.0.13 and B.0.11, we proved that the terms in the second line and third line are simplified to:

$$-\frac{1}{16}\int K\Big(q_{i+1}, \mathrm{ad}_{q_i}\mathrm{ad}_v Bq_{i+2}\Big) = \frac{1}{16}\int K\Big(\mathrm{ad}_v q_{i+1}, \mathrm{ad}_{q_i}Bq_{i+2} - \mathrm{ad}_{q_{i+2}}Bq_i\Big)$$

and

$$\begin{aligned} & -\frac{1}{16}\int K\Big(\mathrm{ad}_v Bq_{i+1}, \mathrm{ad}_{q_i}\mathrm{ad}_v Bq_{i+2}\Big) \\ = & -\frac{1}{32}\int K\Big(\mathrm{ad}_v\mathrm{ad}_v Bq_{i+1} + B\mathrm{ad}_v\mathrm{ad}_v q_{i+1}, \mathrm{ad}_{q_{i+2}}Bq_i - \mathrm{ad}_{q_i}Bq_{i+2}\Big). \end{aligned}$$

The last term, as in the Lemma B.0.12, becomes

$$\begin{aligned} & \frac{1}{4}\int K\Big(D_x B\mathrm{ad}_{q_{i+2}}Bq_i - D_x B\mathrm{ad}_{q_i}Bq_{i+2}, \mathrm{ad}_{D_x v}Bq_{i+1} + D_x q_{i+1}\Big) \\ = & \frac{1}{8}\int K\Big(\mathrm{ad}_{q_{i+2}}Bq_i - \mathrm{ad}_{q_i}Bq_{i+2}, B\mathrm{ad}_v\mathrm{ad}_{D_x v}Bq_{i+1}\Big) \\ + & \frac{1}{16}\int K\Big(\mathrm{ad}_{q_{i+2}}Bq_i - \mathrm{ad}_{q_i}Bq_{i+2}, B\mathrm{ad}_v\mathrm{ad}_v q_{i+1} + \mathrm{ad}_v q_{i+1}\Big) \\ + & \frac{1}{16}\int K\Big(\mathrm{ad}_v Bq_{i+2} + q_{i+2}, \mathrm{ad}_{q_{i+1}}q_i\Big). \end{aligned}$$

This has been proved again in Appendix B. Hence by replacing these equations into the expression for the Schouten bracket we obtain:

$$\begin{aligned} & [\mathfrak{A}^{-1*}S_1\mathfrak{A}^{-1}, \mathfrak{A}^{-1*}S_1\mathfrak{A}^{-1}] \\ = & -\frac{1}{32}\int K\Big(\mathrm{ad}_v\mathrm{ad}_v Bq_{i+1} + B\mathrm{ad}_v\mathrm{ad}_v q_{i+1}, \mathrm{ad}_{q_{i+2}}Bq_i - \mathrm{ad}_{q_i}Bq_{i+2}\Big) \\ + & \frac{1}{8}\int K\Big(\mathrm{ad}_{q_{i+2}}Bq_i - \mathrm{ad}_{q_i}Bq_{i+2}, B\mathrm{ad}_v\mathrm{ad}_{D_x v}Bq_{i+1}\Big) \\ + & \frac{1}{16}\int K\Big(\mathrm{ad}_{q_{i+2}}Bq_i - \mathrm{ad}_{q_i}Bq_{i+2}, B\mathrm{ad}_v\mathrm{ad}_v q_{i+1}\Big) \\ = & \frac{1}{16}\int K\Big(\mathrm{ad}_{q_{i+2}}Bq_i - \mathrm{ad}_{q_i}Bq_{i+2}, \\ & \frac{1}{2}B\mathrm{ad}_v\mathrm{ad}_v q_{i+1} + 2B\mathrm{ad}_v\mathrm{ad}_{D_x v}Bq_{i+1} - \frac{1}{2}\mathrm{ad}_v\mathrm{ad}_v Bq_{i+1}\Big) \end{aligned}$$

Again by using (6.2.3), we can find that

$$\begin{aligned} & \mathrm{ad}_v\mathrm{ad}_v Bq_{i+1} = B(2D_x - \mathrm{ad}_v)\mathrm{ad}_v\mathrm{ad}_v Bq_{i+1} \\ = & 2B\mathrm{ad}_{D_x v}\mathrm{ad}_v Bq_{i+1} + 2B\mathrm{ad}_v\mathrm{ad}_{D_x v}Bq_{i+1} + 2B\mathrm{ad}_v\mathrm{ad}_v D_x Bq_{i+1} \\ - & B\mathrm{ad}_v\mathrm{ad}_v\mathrm{ad}_v Bq_{i+1} \\ = & 2B\mathrm{ad}_{D_x v}\mathrm{ad}_v Bq_{i+1} + 2B\mathrm{ad}_v\mathrm{ad}_{D_x v}Bq_{i+1} \\ + & B\mathrm{ad}_v\mathrm{ad}_v\mathrm{ad}_v Bq_{i+1} + B\mathrm{ad}_v\mathrm{ad}_v q_{i+1} - B\mathrm{ad}_v\mathrm{ad}_v\mathrm{ad}_v Bq_{i+1} \\ = & 2B\mathrm{ad}_{D_x v}\mathrm{ad}_v Bq_{i+1} + 2B\mathrm{ad}_v\mathrm{ad}_{D_x v}Bq_{i+1} + B\mathrm{ad}_v\mathrm{ad}_v q_{i+1}. \end{aligned}$$

Using the last identity in the Schouten bracket, we obtain the following equation:

$$
\begin{aligned}
&[\mathfrak{A}^{-1*}S_1\mathfrak{A}^{-1}, \mathfrak{A}^{-1*}S_1\mathfrak{A}^{-1}] \\
= \ & \frac{1}{16}\int K\Big(\mathrm{ad}_{q_{i+2}}Bq_i - \mathrm{ad}_{q_i}Bq_{i+2}, B\mathrm{ad}_v\mathrm{ad}_{D_xv}Bq_{i+1} - B\mathrm{ad}_{D_xv}\mathrm{ad}_vBq_{i+1}\Big).
\end{aligned}
$$

Now for the second component of the Killing form in the integrand, we can use the Jacobi identity and see that

$$
\begin{aligned}
&[\mathfrak{A}^{-1*}S_1\mathfrak{A}^{-1}, \mathfrak{A}^{-1*}S_1\mathfrak{A}^{-1}] \\
= \ & +\frac{1}{16}\int K\Big(\mathrm{ad}_{q_{i+2}}Bq_i - \mathrm{ad}_{q_i}Bq_{i+2}, -B\mathrm{ad}_{Bq_{i+1}}\mathrm{ad}_vD_xv\Big).
\end{aligned}
$$

We apply Lemma 6.2.5 to simplify the Schouten bracket as follows:

$$
[\mathfrak{A}^{-1*}S_1\mathfrak{A}^{-1}, \mathfrak{A}^{-1*}S_1\mathfrak{A}^{-1}] = \frac{1}{16}\int K\Big(\mathrm{ad}_{Bq_{i+2}}Bq_i, \mathrm{ad}_{Bq_{i+1}}\mathrm{ad}_vD_xv\Big).
$$

Now we take the last step. We use the invariance property of the Killing form under the adjoint action (6.1.4):

$$
[\mathfrak{A}^{-1*}S_1\mathfrak{A}^{-1}, \mathfrak{A}^{-1*}S_1\mathfrak{A}^{-1}] = -\frac{1}{16}\int K\Big(\mathrm{ad}_{Bq_{i+1}}\mathrm{ad}_{Bq_{i+2}}Bq_i, \mathrm{ad}_vD_xv\Big).
$$

This expression is simply zero if we apply the Jacobi identity to the first component of the Killing form. ■

Theorem 6.2.10. The operator $\mathfrak{A}^{-1*}S_2\mathfrak{A}^{-1}$ is symplectic.

Proof. The Schouten bracket of the operator is as follows:

$$
\begin{aligned}
&[\mathfrak{A}^{-1*}S_2\mathfrak{A}^{-1}, \mathfrak{A}^{-1*}S_2\mathfrak{A}^{-1}] = \int K(p_i, D_{\hat{\mathfrak{A}}^{-1*}S_2\hat{\mathfrak{A}}^{-1}}[p_{i+2}]p_{i+1}) \\
= \ & \int K\Big(\mathbf{q}_i, D_x\mathrm{ad}_{\mathbf{q}_{i+2}}Bq_{i+1} + D_x\mathrm{ad}_{\mathbf{v}}B\mathrm{ad}_{q_{i+2}}Bq_{i+1}\Big) \\
+ \ & \int K\Big(\mathbf{q}_i, -\frac{1}{2}\mathrm{ad}_{\mathbf{q}_{i+2}}D_xBq_{i+1} - \frac{1}{2}\mathrm{ad}_{\mathbf{v}}D_xB\mathrm{ad}_{q_{i+2}}Bq_{i+1}\Big) \\
+ \ & \int K\Big(q_i, \frac{1}{2}B\mathrm{ad}_{q_{i+2}}B\pi_1\mathrm{ad}_{\mathrm{ad}_{\mathbf{v}}D_x\mathbf{v}}Bq_{i+1} + \frac{1}{2}B\pi_1\mathrm{ad}_{\mathrm{ad}_{\mathbf{q}_{i+2}}D_x\mathbf{v}}Bq_{i+1}\Big) \\
+ \ & \int K\Big(q_i, \frac{1}{2}B\pi_1\mathrm{ad}_{\mathrm{ad}_{\mathbf{v}}D_x\mathbf{q}_{i+2}}Bq_{i+1} + \frac{1}{2}B\pi_1\mathrm{ad}_{\mathrm{ad}_{\mathbf{v}}D_x\mathbf{v}}B\mathrm{ad}_{q_{i+2}}Bq_{i+1}\Big) \\
+ \ & \int K\Big(q_i, B\mathrm{ad}_{q_{i+2}}B\pi_1\mathrm{ad}_{\mathbf{v}}D_x\mathbf{q}_{i+1} + B\pi_1\mathrm{ad}_{\mathbf{q}_{i+2}}D_x\mathbf{q}_{i+1}\Big) \\
+ \ & \int K\Big(q_i, -\frac{1}{2}B\mathrm{ad}_{q_{i+2}}BD_x\pi_1\mathrm{ad}_{\mathbf{v}}\mathbf{q}_{i+1} - \frac{1}{2}BD_x\pi_1\mathrm{ad}_{\mathbf{q}_{i+2}}\mathbf{q}_{i+1}\Big).
\end{aligned}
$$

Those subexpressions which are compatible are listed below and it is proved that each of them cancels. The first of these three subexpressions is:

$$\begin{aligned}
- & \frac{1}{2}\int K\Big(\mathbf{q}_i, \mathrm{ad}_{\mathbf{q}_{i+2}} D_x B q_{i+1}\Big) - \frac{1}{2}\int K\Big(q_i, B D_x \pi_1 \mathrm{ad}_{\mathbf{q}_{i+2}} \mathbf{q}_{i+1}\Big) \\
+ & \int K\Big(\mathbf{q}_i, D_x \mathrm{ad}_{\mathbf{q}_{i+2}} B q_{i+1}\Big) + \int K\Big(q_i, B \pi_1 \mathrm{ad}_{\mathbf{q}_{i+2}} D_x \mathbf{q}_{i+1}\Big) \\
= & -\frac{1}{2}\int K\Big(D_x \mathrm{ad}_{\mathbf{q}_{i+2}} \mathbf{q}_i, B q_{i+1}\Big) + \frac{1}{2}\int K\Big(B q_{i+1}, D_x \pi_1 \mathrm{ad}_{\mathbf{q}_i} \mathbf{q}_{i+2}\Big) \\
+ & \int K\Big(\mathrm{ad}_{\mathbf{q}_{i+2}} D_x \mathbf{q}_i, B q_{i+1}\Big) - \int K\Big(B q_{i+1}, \pi_1 \mathrm{ad}_{\mathbf{q}_i} D_x \mathbf{q}_{i+2}\Big) \\
= & -\int K\Big(D_x \mathrm{ad}_{\mathbf{q}_{i+2}} \mathbf{q}_i, B q_{i+1}\Big) \\
+ & \int K\Big(\mathrm{ad}_{\mathbf{q}_{i+2}} D_x \mathbf{q}_i, B q_{i+1}\Big) + \int K\Big(B q_{i+1}, \pi_1 \mathrm{ad}_{D_x \mathbf{q}_{i+2}} \mathbf{q}_i\Big) \\
= & \int K\Big(- D_x \mathrm{ad}_{\mathbf{q}_{i+2}} \mathbf{q}_i + \mathrm{ad}_{\mathbf{q}_{i+2}} D_x \mathbf{q}_i + \mathrm{ad}_{D_x \mathbf{q}_{i+2}} \mathbf{q}_i, B q_{i+1}\Big) \\
= & \; 0.
\end{aligned}$$

This has been done by just applying the rules used so far. For instance, in the first line we used invariance of the Killing form (6.1.4) and integration by parts, and also the fact that the operator B is anti-symmetric.

The second subexpression is:

$$\frac{1}{2}\int K(q_i, B\mathrm{ad}_{q_{i+2}} B \pi_1 \mathrm{ad}_{\mathrm{ad}_{\mathbf{v}} D_x \mathbf{v}} B q_{i+1}) + K(q_i, B \pi_1 \mathrm{ad}_{\mathrm{ad}_{\mathbf{v}} D_x \mathbf{v}} B \mathrm{ad}_{q_{i+2}} B q_{i+1})$$

The first term of the expression is simplified to

$$-\frac{1}{4}\int K\Big(B\mathrm{ad}_{q_{i+2}} B q_i, \pi_1 \mathrm{ad}_{\mathrm{ad}_{\mathbf{v}} D_x \mathbf{v}} B q_{i+1}\Big)$$

using $B^* = -B$ and the invariant property (6.1.4). The second term similarly becomes

$$\frac{1}{2}\int K\Big(\mathrm{ad}_{\mathrm{ad}_{\mathbf{v}} D_x \mathbf{v}} B q_i, B \mathrm{ad}_{q_{i+2}} B q_{i+1}\Big),$$

using the same rules. Then if one applies the shifting rule, we obtain

$$\frac{1}{2}\int K\Big(\mathrm{ad}_{\mathrm{ad}_{\mathbf{v}} D_x \mathbf{v}} B q_{i+1}, B \mathrm{ad}_{q_i} B q_{i+2}\Big).$$

Hence the couple simplifies to:

$$-\frac{1}{2}\int K\Big(B\mathrm{ad}_{q_{i+2}} B q_i - B\mathrm{ad}_{q_i} B q_{i+2}, \mathrm{ad}_{\mathrm{ad}_{\mathbf{v}} D_x \mathbf{v}} B q_{i+1}\Big).$$

Now if we apply Lemma 6.2.5 we obtain the following expression:

$$-\frac{1}{2}\int K\Big(\mathrm{ad}_{Bq_{i+2}}Bq_i, \mathrm{ad}_{\mathrm{ad}_{\mathbf{v}}D_x\mathbf{v}}Bq_{i+1}\Big).$$

Again using the invariance rule, we obtain:

$$-\frac{1}{2}\int K\Big(\mathrm{ad}_{Bq_{i+1}}\mathrm{ad}_{Bq_{i+2}}Bq_i, \mathrm{ad}_{\mathbf{v}}D_x\mathbf{v}\Big).$$

The last term is zero because of the Jacobi identity.

The last subexpression is:

$$\begin{aligned}
& \frac{1}{2}\int K\Big(q_i, B\pi_1\mathrm{ad}_{\mathrm{ad}_{\mathbf{q}_{i+2}}D_x\mathbf{v}}Bq_{i+1}\Big) + \frac{1}{2}\int K\Big(q_i, B\pi_1\mathrm{ad}_{\mathrm{ad}_{\mathbf{v}}D_x\mathbf{q}_{i+2}}Bq_{i+1}\Big)\\
- \; & \frac{1}{2}\int K\Big(\mathbf{q}_i, \mathrm{ad}_{\mathbf{v}}D_xB\mathrm{ad}_{q_{i+2}}Bq_{i+1}\Big) - \frac{1}{2}\int K\Big(q_i, B\mathrm{ad}_{q_{i+2}}BD_x\pi_1\mathrm{ad}_{\mathbf{v}}\mathbf{q}_{i+1}\Big)\\
+ \; & \int K\Big(\mathbf{q}_i, D_x\mathrm{ad}_{\mathbf{v}}B\mathrm{ad}_{q_{i+2}}Bq_{i+1}\Big) + \int K\Big(q_i, B\mathrm{ad}_{q_{i+2}}B\pi_1\mathrm{ad}_{\mathbf{v}}D_x\mathbf{q}_{i+1}\Big).
\end{aligned}$$

Applying the rules just mentioned, the expression becomes as follows:

$$\begin{aligned}
- \; & \frac{1}{2}\int K\Big(\mathrm{ad}_{Bq_{i+1}}Bq_i, \mathrm{ad}_{\mathbf{q}_{i+2}}D_x\mathbf{v}\Big) - \int K\Big(\mathrm{ad}_{Bq_{i+1}}Bq_i, \mathrm{ad}_{\mathbf{v}}D_x\mathbf{q}_{i+2}\Big)\\
- \; & \frac{1}{2}\int K\Big(D_x\mathrm{ad}_{\mathbf{v}}\mathbf{q}_i, B\mathrm{ad}_{q_{i+2}}Bq_{i+1}\Big) + \frac{1}{2}\int K\Big(B\mathrm{ad}_{q_{i+2}}Bq_i, D_x\pi_1\mathrm{ad}_{\mathbf{v}}\mathbf{q}_{i+1}\Big)\\
+ \; & \int K\Big(\mathrm{ad}_{\mathbf{v}}D_x\mathbf{q}_i, B\mathrm{ad}_{q_{i+2}}Bq_{i+1}\Big) - \int K\Big(B\mathrm{ad}_{q_{i+2}}Bq_i, \pi_1\mathrm{ad}_{\mathbf{v}}D_x\mathbf{q}_{i+1}\Big).
\end{aligned}$$

Now, using Lemma 6.2.5, the first line becomes

$$-\frac{1}{2}\int K\Big(\mathrm{ad}_{Bq_{i+1}}Bq_i, \mathrm{ad}_{\mathbf{q}_{i+2}}D_x\mathbf{v} + \mathrm{ad}_{\mathbf{v}}D_x\mathbf{q}_{i+2}\Big).$$

Applying the shifting rule, the first line equals

$$\int K\Big(\mathrm{ad}_{\mathbf{v}}D_x\mathbf{q}_{i+1}, B\mathrm{ad}_{q_i}Bq_{i+2} - B\mathrm{ad}_{q_{i+2}}Bq_i\Big),$$

So that the the whole expression takes the form

$$\begin{aligned}
- \; & \frac{1}{2}\int K\Big(B\mathrm{ad}_{q_i}Bq_{i+2} - B\mathrm{ad}_{q_{i+2}}Bq_i, \mathrm{ad}_{\mathbf{q}_{i+1}}D_x\mathbf{v} + \mathrm{ad}_{\mathbf{v}}D_x\mathbf{q}_{i+1}\Big)\\
- \; & \frac{1}{2}\int K\Big(D_x\mathrm{ad}_{\mathbf{v}}\mathbf{q}_{i+1}, B\mathrm{ad}_{q_i}Bq_{i+2} - B\mathrm{ad}_{q_{i+2}}Bq_i\Big)\\
+ \; & \int K\Big(\mathrm{ad}_{\mathbf{v}}D_x\mathbf{q}_{i+1}, B\mathrm{ad}_{q_i}Bq_{i+2} - B\mathrm{ad}_{q_{i+2}}Bq_i\Big)\\
= \; & \frac{1}{2}\int K\Big(-D_x\mathrm{ad}_{\mathbf{v}}\mathbf{q}_{i+1} + \mathrm{ad}_{\mathbf{v}}D_x\mathbf{q}_{i+1} + \mathrm{ad}_{D_x\mathbf{v}}\mathbf{q}_{i+1}, B\mathrm{ad}_{q_i}Bq_{i+2} - B\mathrm{ad}_{q_{i+2}}Bq_i\Big).
\end{aligned}$$

The last expression is zero, since D_x is a derivation. Hence it has been proved that the Schouten bracket

$$[\hat{\mathfrak{A}}^{-1*}S_2\hat{\mathfrak{A}}^{-1}, \hat{\mathfrak{A}}^{-1*}S_2\hat{\mathfrak{A}}^{-1}]$$

vanishes and this shows that the operator $\hat{\mathfrak{A}}^{-1*}S_2\hat{\mathfrak{A}}^{-1}$ is a symplectic operator. ■