Integrable Systems and Symplectic Geometry

Asadi, E.

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Appendix A

Computations in Hamiltonian operators

In this appendix, we prove the following identity for an anti-symmetric operator $H : \mathfrak{E}^* \to \mathfrak{E}$:

$$d^2\omega_H(\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3) = [H, H](\mathfrak{b}_1^*, \mathfrak{b}_3^*, \mathfrak{b}_2^*),$$

where $[H, H]$ is the Schouten or Schouten bracket of H and $\mathfrak{a}_i = H\mathfrak{b}_i^*$ for $\mathfrak{b}_i^* \in \mathfrak{E}^*$. We compute the left hand side of the identity. Using the formula for the coboundary operator d^2 as in (2.1.4) or (2.1.1), we have that

$$\begin{aligned}
& d^2\omega_H(\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3) \\
= \; & \mathfrak{a}_1.\omega_H(\mathfrak{a}_2, \mathfrak{a}_3) - \mathfrak{a}_2.\omega_H(\mathfrak{a}_1, \mathfrak{a}_3) + \mathfrak{a}_3.\omega_H(\mathfrak{a}_1, \mathfrak{a}_2) \\
& -\omega_H([\mathfrak{a}_1, \mathfrak{a}_2], \mathfrak{a}_3) + \omega_H([\mathfrak{a}_1, \mathfrak{a}_3], \mathfrak{a}_2) - \omega_H([\mathfrak{a}_2, \mathfrak{a}_3], \mathfrak{a}_1) \\
= \; & \mathfrak{a}_1.\omega_H(\mathfrak{a}_2, \mathfrak{a}_3) + \mathfrak{a}_2.\omega_H(\mathfrak{a}_3, \mathfrak{a}_1) + \mathfrak{a}_3.\omega_H(\mathfrak{a}_1, \mathfrak{a}_2) \\
& -\omega_H([\mathfrak{a}_1, \mathfrak{a}_2], \mathfrak{a}_3) + \omega_H([\mathfrak{a}_1, \mathfrak{a}_3], \mathfrak{a}_2) - \omega_H([\mathfrak{a}_2, \mathfrak{a}_3], \mathfrak{a}_1).
\end{aligned}$$

Hence using Definition 2.1.5 of the 2-form ω_H, we find that

$$\begin{aligned}
& d^2\omega_H(\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3) \\
= \; & L_{\mathfrak{a}_1}(\mathfrak{b}_3^*(\mathfrak{a}_2)) + L_{\mathfrak{a}_2}(\mathfrak{b}_1^*(\mathfrak{a}_3)) + L_{\mathfrak{a}_3}(\mathfrak{b}_2^*(\mathfrak{a}_1)) \\
& -\mathfrak{b}_3^*([\mathfrak{a}_1, \mathfrak{a}_2]) + \mathfrak{b}_2^*([\mathfrak{a}_1, \mathfrak{a}_3]) - \mathfrak{b}_1^*([\mathfrak{a}_2, \mathfrak{a}_3]).
\end{aligned}$$

Using (2.1.3), we see that

$$\begin{aligned}
& d^2\omega_H(\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3) \\
= \; & \mathfrak{b}_3^*(L_{\mathfrak{a}_1}\mathfrak{a}_2) + (L_{\mathfrak{a}_1}\mathfrak{b}_3^*)(\mathfrak{a}_2) \\
& \mathfrak{b}_1^*(L_{\mathfrak{a}_2}\mathfrak{a}_3) + (L_{\mathfrak{a}_2}\mathfrak{b}_1^*)(\mathfrak{a}_3) \\
& \mathfrak{b}_2^*(L_{\mathfrak{a}_3}\mathfrak{a}_1) + (L_{\mathfrak{a}_3}\mathfrak{b}_2^*)(\mathfrak{a}_1) \\
& -\mathfrak{b}_3^*([\mathfrak{a}_1, \mathfrak{a}_2]) + \mathfrak{b}_2^*([\mathfrak{a}_1, \mathfrak{a}_3]) - \mathfrak{b}_1^*([\mathfrak{a}_2, \mathfrak{a}_3]) \\
= \; & (L_{\mathfrak{a}_1}\mathfrak{b}_3^*)(\mathfrak{a}_2) + (L_{\mathfrak{a}_2}\mathfrak{b}_1^*)(\mathfrak{a}_3) + (L_{\mathfrak{a}_3}\mathfrak{b}_2^*)(\mathfrak{a}_1) \\
= \; & (L_{H\mathfrak{b}_1^*}\mathfrak{b}_3^*)(H\mathfrak{b}_2^*) + (L_{H\mathfrak{b}_2^*}\mathfrak{b}_1^*)(H\mathfrak{b}_3^*) + (L_{H\mathfrak{b}_3^*}\mathfrak{b}_2^*)(H\mathfrak{b}_1^*).
\end{aligned}$$

According to Definition 2.3.1, this shows that

$$d^2\omega_H(\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3) = [H, H](\mathfrak{b}_1^*, \mathfrak{b}_3^*, \mathfrak{b}_2^*).$$