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2008

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citation for published version (APA)

Asadi, E. (2008). *Integrable Systems and Symplectic Geometry*.

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Appendix B

Some Killing form identities

Justification of some equalities used in Theorem 6.2.9 did not fit there to include. Here we prove that those equalities indeed holds.

Lemma B.0.11. Let B be the antisymmetric operator $B = (2D_x - \text{ad}_v)^{-1}$ acting on purely imaginary (in the quaternion sense) functions. Then

$$\begin{aligned} & \sum_{i \in \mathbb{Z}/3} \int K\left(\text{ad}_v Bq_{i+1}, \text{ad}_{q_i} \text{ad}_v Bq_{i+2}\right) \\ &= \frac{1}{2} \sum_{i \in \mathbb{Z}/3} \int K\left(\text{ad}_v \text{ad}_v Bq_{i+1} + B \text{ad}_v \text{ad}_v q_{i+1}, \text{ad}_{q_{i+2}} Bq_i - \text{ad}_{q_i} Bq_{i+2}\right). \end{aligned}$$

Here we explicitly sum over $i \in \mathbb{Z}/3$, contrary to the convention of Notation 6.1.1.

Proof. The integral on the left hand side of the equality can be expanded as below using the invariance of the Killing form K and Jacobi identity and sorting out the expressions afterwards.

$$\begin{aligned} & \int K\left(\text{ad}_v Bq_{i+1}, \text{ad}_{q_i} \text{ad}_v Bq_{i+2}\right) \\ &= \frac{1}{2} \int K\left(\text{ad}_v Bq_{i+1}, \text{ad}_{q_i} \text{ad}_v Bq_{i+2}\right) + \frac{1}{2} \int K\left(\text{ad}_v Bq_{i+1}, \text{ad}_{q_i} \text{ad}_v Bq_{i+2}\right) \\ &= \frac{1}{2} \int K\left(\text{ad}_v Bq_{i+1}, \text{ad}_{q_i} \text{ad}_v Bq_{i+2}\right) - \frac{1}{2} \int K\left(\text{ad}_{q_i} \text{ad}_v Bq_{i+1}, \text{ad}_v Bq_{i+2}\right) \\ &= \frac{1}{2} \int K\left(\text{ad}_v Bq_{i+1}, -\text{ad}_v \text{ad}_{Bq_{i+2}} q_i - \text{ad}_{Bq_{i+2}} \text{ad}_{q_i} v\right) \\ &\quad - \frac{1}{2} \int K\left(-\text{ad}_v \text{ad}_{Bq_{i+1}} q_i - \text{ad}_{Bq_{i+1}} \text{ad}_{q_i} v, \text{ad}_v Bq_{i+2}\right) \\ &= \frac{1}{2} \int K\left(\text{ad}_v \text{ad}_{Bq_{i+1}} q_i, \text{ad}_v Bq_{i+2}\right) - \frac{1}{2} \int K\left(\text{ad}_v Bq_{i+1}, \text{ad}_v \text{ad}_{Bq_{i+2}} q_i\right) \\ &+ \frac{1}{2} \int K\left(\text{ad}_{Bq_{i+1}} \text{ad}_{q_i} v, \text{ad}_v Bq_{i+2}\right) - \frac{1}{2} \int K\left(\text{ad}_v Bq_{i+1}, \text{ad}_{Bq_{i+2}} \text{ad}_{q_i} v\right) \end{aligned}$$

Now the expression becomes as follows using the invariance of the Killing.

$$\begin{aligned}
& \int K\left(\operatorname{ad}_v Bq_{i+1}, \operatorname{ad}_{q_i} \operatorname{ad}_v Bq_{i+2}\right) \\
&= \frac{1}{2} \int K\left(\operatorname{ad}_v \operatorname{ad}_v Bq_{i+1}, \operatorname{ad}_{Bq_{i+2}} q_i\right) - \frac{1}{2} \int K\left(\operatorname{ad}_v \operatorname{ad}_v \operatorname{ad}_{Bq_{i+1}} q_i, Bq_{i+2}\right) \\
&+ \frac{1}{2} \int K\left(\operatorname{ad}_{Bq_{i+2}} \operatorname{ad}_v Bq_{i+1}, \operatorname{ad}_{q_i} v\right) - \frac{1}{2} \int K\left(\operatorname{ad}_{Bq_{i+1}} \operatorname{ad}_v Bq_{i+2}, \operatorname{ad}_{q_i} v\right) \\
&= -\frac{1}{2} \int K\left(\operatorname{ad}_v \operatorname{ad}_v Bq_{i+1}, \operatorname{ad}_{q_i} Bq_{i+2}\right) - \frac{1}{2} \int K\left(\operatorname{ad}_{Bq_{i+1}} q_i, \operatorname{ad}_v \operatorname{ad}_v Bq_{i+2}\right) \\
&+ \frac{1}{2} \int K\left(\operatorname{ad}_{Bq_{i+2}} \operatorname{ad}_v Bq_{i+1}, \operatorname{ad}_{q_i} v\right) - \frac{1}{2} \int K\left(\operatorname{ad}_{Bq_{i+1}} \operatorname{ad}_v Bq_{i+2}, \operatorname{ad}_{q_i} v\right)
\end{aligned}$$

It follows that

$$\begin{aligned}
& \sum_{i \in \mathbb{Z}/3} \int K\left(\operatorname{ad}_v Bq_{i+1}, \operatorname{ad}_{q_i} \operatorname{ad}_v Bq_{i+2}\right) \\
&= \frac{1}{2} \sum_{i \in \mathbb{Z}/3} \int K\left(\operatorname{ad}_v \operatorname{ad}_v Bq_{i+1}, \operatorname{ad}_{q_{i+2}} Bq_i - \operatorname{ad}_{q_i} Bq_{i+2}\right) - K\left(\operatorname{ad}_v \operatorname{ad}_{Bq_{i+1}} Bq_{i+2}, \operatorname{ad}_{q_i} v\right) \\
&= \frac{1}{2} \sum_{i \in \mathbb{Z}/3} \int K\left(\operatorname{ad}_v \operatorname{ad}_v Bq_{i+1}, \operatorname{ad}_{q_{i+2}} Bq_i - \operatorname{ad}_{q_i} Bq_{i+2}\right) + K\left(\operatorname{ad}_{Bq_{i+1}} Bq_{i+2}, \operatorname{ad}_v \operatorname{ad}_{q_i} v\right)
\end{aligned}$$

It follows from Lemma 6.2.5 that the following identity holds for the operator B :

$$\int K\left(\operatorname{ad}_{Bq_{i+1}} Bq_{i+2}, \cdot\right) = \int K\left(B \operatorname{ad}_{q_{i+1}} Bq_{i+2} - B \operatorname{ad}_{q_{i+2}} Bq_{i+1}, \cdot\right).$$

We find that

$$\begin{aligned}
& \sum_{i \in \mathbb{Z}/3} \int K\left(\operatorname{ad}_v Bq_{i+1}, \operatorname{ad}_{q_i} \operatorname{ad}_v Bq_{i+2}\right) \\
&= \frac{1}{2} \sum_{i \in \mathbb{Z}/3} \int K\left(\operatorname{ad}_v \operatorname{ad}_v Bq_{i+1}, \operatorname{ad}_{q_{i+2}} Bq_i - \operatorname{ad}_{q_i} Bq_{i+2}\right) \\
&- \frac{1}{2} \sum_{i \in \mathbb{Z}/3} \int K\left(B \operatorname{ad}_v \operatorname{ad}_{q_{i+1}} v, \operatorname{ad}_{q_{i+2}} Bq_i - \operatorname{ad}_{q_i} Bq_{i+2}\right). \\
&= \frac{1}{2} \sum_{i \in \mathbb{Z}/3} \int K\left(\operatorname{ad}_v \operatorname{ad}_v Bq_{i+1} - B \operatorname{ad}_v \operatorname{ad}_{q_{i+1}} v, \operatorname{ad}_{q_{i+2}} Bq_i - \operatorname{ad}_{q_i} Bq_{i+2}\right) \\
&= \frac{1}{2} \sum_{i \in \mathbb{Z}/3} \int K\left(\operatorname{ad}_v \operatorname{ad}_v Bq_{i+1} + B \operatorname{ad}_v \operatorname{ad}_v q_{i+1}, \operatorname{ad}_{q_{i+2}} Bq_i - \operatorname{ad}_{q_i} Bq_{i+2}\right)
\end{aligned}$$

■

Lemma B.0.12.

$$\begin{aligned}
& \int K\left(D_x \text{Bad}_{q_{i+2}} Bq_i - D_x \text{Bad}_{q_i} Bq_{i+2}, \text{ad}_{D_x v} Bq_{i+1} + D_x q_{i+1}\right) \\
&= \frac{1}{2} \int K\left(\text{ad}_{q_{i+2}} Bq_i - \text{ad}_{q_i} Bq_{i+2}, \text{Bad}_v \text{ad}_{D_x v} Bq_{i+1}\right) \\
&+ \frac{1}{4} \int K\left(\text{ad}_{q_{i+2}} Bq_i - \text{ad}_{q_i} Bq_{i+2}, \text{Bad}_v \text{ad}_v q_{i+1} + \text{ad}_v q_{i+1}\right) \\
&+ \frac{1}{4} \int K\left(\text{ad}_v Bq_{i+2} + q_{i+2}, \text{ad}_{q_{i+1}} q_i\right).
\end{aligned}$$

Proof. Remember that $D_x B = \frac{1}{2} + \frac{1}{2} \text{ad}_v B$.

$$\begin{aligned}
& \int K\left(D_x \text{Bad}_{q_{i+2}} Bq_i - D_x \text{Bad}_{q_i} Bq_{i+2}, \text{ad}_{D_x v} Bq_{i+1} + D_x q_{i+1}\right) \\
&= \frac{1}{2} \int K\left(\text{ad}_v \text{Bad}_{q_{i+2}} Bq_i - \text{ad}_v \text{Bad}_{q_i} Bq_{i+2}, \text{ad}_{D_x v} Bq_{i+1} + D_x q_{i+1}\right) \\
&+ \frac{1}{2} \int K\left(\text{ad}_{q_{i+2}} Bq_i - \text{ad}_{q_i} Bq_{i+2}, \text{ad}_{D_x v} Bq_{i+1} + D_x q_{i+1}\right)
\end{aligned}$$

and it follows that

$$\begin{aligned}
& \sum_{i \in \mathbb{Z}/3} \int K\left(D_x \text{Bad}_{q_{i+2}} Bq_i - D_x \text{Bad}_{q_i} Bq_{i+2}, \text{ad}_{D_x v} Bq_{i+1} + D_x q_{i+1}\right) \\
&= \frac{1}{2} \sum_{i \in \mathbb{Z}/3} \int K\left(\text{ad}_v \text{Bad}_{q_{i+2}} Bq_i - \text{ad}_v \text{Bad}_{q_i} Bq_{i+2}, \text{ad}_{D_x v} Bq_{i+1} + D_x q_{i+1}\right) \\
&- \frac{1}{2} \sum_{i \in \mathbb{Z}/3} \int K\left(\text{ad}_{D_x v} \text{ad}_{q_{i+2}} Bq_i, Bq_{i+1}\right) + \frac{1}{2} \sum_{i \in \mathbb{Z}/3} \int K\left(Bq_{i+2}, \text{ad}_{q_i} \text{ad}_{D_x v} Bq_{i+1}\right) \\
&- \frac{1}{2} \sum_{i \in \mathbb{Z}/3} \int K\left(Bq_i, \text{ad}_{q_{i+2}} D_x q_{i+1}\right) + \frac{1}{2} \sum_{i \in \mathbb{Z}/3} \int K\left(Bq_{i+2}, \text{ad}_{q_i} D_x q_{i+1}\right) \\
&= \frac{1}{2} \sum_{i \in \mathbb{Z}/3} \int K\left(\text{ad}_v \text{Bad}_{q_{i+2}} Bq_i - \text{ad}_v \text{Bad}_{q_i} Bq_{i+2}, \text{ad}_{D_x v} Bq_{i+1} + D_x q_{i+1}\right) \\
&- \frac{1}{2} \sum_{i \in \mathbb{Z}/3} \int K\left(\text{ad}_{D_x v} \text{ad}_{q_i} Bq_{i+1}, Bq_{i+2}\right) + \frac{1}{2} \sum_{i \in \mathbb{Z}/3} \int K\left(Bq_{i+2}, \text{ad}_{q_i} \text{ad}_{D_x v} Bq_{i+1}\right) \\
&- \frac{1}{2} \sum_{i \in \mathbb{Z}/3} \int K\left(Bq_{i+2}, \text{ad}_{q_{i+1}} D_x q_i\right) + \frac{1}{2} \sum_{i \in \mathbb{Z}/3} \int K\left(Bq_{i+2}, \text{ad}_{q_i} D_x q_{i+1}\right) \\
&= \frac{1}{2} \sum_{i \in \mathbb{Z}/3} \int K\left(\text{ad}_v \text{Bad}_{q_{i+2}} Bq_i - \text{ad}_v \text{Bad}_{q_i} Bq_{i+2}, \text{ad}_{D_x v} Bq_{i+1} + D_x q_{i+1}\right) \\
&- \frac{1}{2} \sum_{i \in \mathbb{Z}/3} \int K\left(\text{ad}_{Bq_{i+1}} \text{ad}_{q_i} D_x v, Bq_{i+2}\right) - \frac{1}{2} \sum_{i \in \mathbb{Z}/3} \int K\left(Bq_{i+2}, D_x \text{ad}_{q_{i+1}} q_i\right)
\end{aligned}$$

where the last two terms were obtained by applying the Jacobi identity and the derivation property. We now have, using again Lemma 6.2.5,

$$\begin{aligned}
& \sum_{i \in \mathbb{Z}/3} \int K \left(D_x \text{Bad}_{q_{i+2}} Bq_i - D_x \text{Bad}_{q_i} Bq_{i+2}, \text{ad}_{D_x v} Bq_{i+1} + D_x q_{i+1} \right) \\
&= \frac{1}{2} \sum_{i \in \mathbb{Z}/3} \int K \left(\text{ad}_v \text{Bad}_{q_{i+2}} Bq_i - \text{ad}_v \text{Bad}_{q_i} Bq_{i+2}, \text{ad}_{D_x v} Bq_{i+1} + D_x q_{i+1} \right) \\
&+ \frac{1}{2} \sum_{i \in \mathbb{Z}/3} \int K \left(\text{ad}_{Bq_{i+1}} Bq_{i+2}, \text{ad}_{q_i} D_x v \right) - \frac{1}{2} \sum_{i \in \mathbb{Z}/3} \int K \left(Bq_{i+2}, D_x \text{ad}_{q_{i+1}} q_i \right) \\
&= \frac{1}{2} \sum_{i \in \mathbb{Z}/3} \int K \left(\text{ad}_{q_{i+1}} Bq_{i+2} - \text{ad}_{q_{i+2}} Bq_{i+1} \right. \\
&\quad \left. , \text{Bad}_v \text{ad}_{D_x v} Bq_i + \text{Bad}_v D_x q_i - \text{Bad}_{q_i} D_x v \right) \\
&+ \frac{1}{4} \sum_{i \in \mathbb{Z}/3} \int K \left(\text{ad}_v Bq_{i+2} + q_{i+2}, \text{ad}_{q_{i+1}} q_i \right) \\
&= \frac{1}{2} \sum_{i \in \mathbb{Z}/3} \int K \left(\text{ad}_{q_{i+1}} Bq_{i+2} - \text{ad}_{q_{i+2}} Bq_{i+1}, \text{Bad}_v \text{ad}_{D_x v} Bq_i + B D_x \text{ad}_v q_i \right) \\
&+ \frac{1}{4} \sum_{i \in \mathbb{Z}/3} \int K \left(\text{ad}_v Bq_{i+2} + q_{i+2}, \text{ad}_{q_{i+1}} q_i \right) \\
&= \frac{1}{2} \sum_{i \in \mathbb{Z}/3} \int K \left(\text{ad}_{q_{i+1}} Bq_{i+2} - \text{ad}_{q_{i+2}} Bq_{i+1}, \text{Bad}_v \text{ad}_{D_x v} Bq_i \right) \\
&+ \frac{1}{4} \sum_{i \in \mathbb{Z}/3} \int K \left(\text{ad}_{q_{i+1}} Bq_{i+2} - \text{ad}_{q_{i+2}} Bq_{i+1}, \text{ad}_v q_i + \text{Bad}_v \text{ad}_v q_i \right) \\
&+ \frac{1}{4} \sum_{i \in \mathbb{Z}/3} \int K \left(\text{ad}_v Bq_{i+2} + q_{i+2}, \text{ad}_{q_{i+1}} q_i \right) \\
&= \frac{1}{2} \sum_{i \in \mathbb{Z}/3} \int K \left(\text{ad}_{q_{i+2}} Bq_i - \text{ad}_{q_i} Bq_{i+2}, \text{Bad}_v \text{ad}_{D_x v} Bq_{i+1} \right) \\
&+ \frac{1}{4} \sum_{i \in \mathbb{Z}/3} \int K \left(\text{ad}_{q_{i+2}} Bq_i - \text{ad}_{q_i} Bq_{i+2}, \text{ad}_v q_{i+1} + \text{Bad}_v \text{ad}_v q_{i+1} \right) \\
&+ \frac{1}{4} \sum_{i \in \mathbb{Z}/3} \int K \left(\text{ad}_v Bq_{i+2} + q_{i+2}, \text{ad}_{q_{i+1}} q_i \right)
\end{aligned}$$

where we used the relation $B D_x = \frac{1}{2} + \frac{1}{2} \text{Bad}_v$. ■

Lemma B.0.13.

$$\begin{aligned} & \sum_{i \in \mathbb{Z}/3} \int K(q_{i+1}, \text{ad}_{q_i} \text{ad}_v Bq_{i+2}) \\ &= - \sum_{i \in \mathbb{Z}/3} \int K(\text{ad}_v q_{i+1}, \text{ad}_{q_i} Bq_{i+2} - \text{ad}_{q_{i+2}} Bq_i). \end{aligned}$$

Proof.

$$\begin{aligned} & \sum_{i \in \mathbb{Z}/3} \int K(q_{i+1}, \text{ad}_{q_i} \text{ad}_v Bq_{i+2}) \\ &= - \sum_{i \in \mathbb{Z}/3} \int K(q_{i+1}, \text{ad}_v \text{ad}_{Bq_{i+2}} q_i) - \sum_{i \in \mathbb{Z}/3} \int K(q_{i+1}, \text{ad}_{B_{i+2}} \text{ad}_{q_i} v) \\ &= + \sum_{i \in \mathbb{Z}/3} \int K(\text{ad}_v q_{i+1}, \text{ad}_{Bq_{i+2}} q_i) + \sum_{i \in \mathbb{Z}/3} \int K(\text{ad}_{Bq_{i+2}} q_{i+1}, \text{ad}_{q_i} v) \\ &= - \sum_{i \in \mathbb{Z}/3} \int K(\text{ad}_v q_{i+1}, \text{ad}_{q_i} Bq_{i+2}) - \sum_{i \in \mathbb{Z}/3} \int K(\text{ad}_{q_{i+2}} Bq_i, \text{ad}_{q_{i+1}} v) \\ &= - \sum_{i \in \mathbb{Z}/3} \int K(\text{ad}_v q_{i+1}, \text{ad}_{q_i} Bq_{i+2} - \text{ad}_{q_{i+2}} Bq_i). \end{aligned}$$

■

