Integrable Systems and Symplectic Geometry

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Appendix B

Some Killing form identities

Justification of some equlities used in Theorem 6.2.9 did not fit there to include. Here we prove that those equlities indeed holds.

Lemma B.0.11. Let B be the antisymmetric operator $B = (2D_x - \mathrm{ad}_v)^{-1}$ acting on purely imaginary (in the quaternion sense) functions. Then

$$
\begin{aligned}
& \sum_{i\in\mathbb{Z}/3} \int K\Big(\mathrm{ad}_v Bq_{i+1}, \mathrm{ad}_{q_i}\mathrm{ad}_v Bq_{i+2}\Big) \\
= \; & \frac{1}{2}\sum_{i\in\mathbb{Z}/3} \int K\Big(\mathrm{ad}_v\mathrm{ad}_v Bq_{i+1} + B\mathrm{ad}_v\mathrm{ad}_v q_{i+1}, \mathrm{ad}_{q_{i+2}}Bq_i - \mathrm{ad}_{q_i}Bq_{i+2}\Big).
\end{aligned}
$$

Here we explicitly sum over $i \in \mathbb{Z}/3$, contrary to the convention of Notation 6.1.1.

Proof. The integral on the left hand side of the equality can be expanded as below using the invariance of the Killing form K and Jacobi identity and sorting out the expressions afterwards.

$$
\begin{aligned}
& \int K\Big(\mathrm{ad}_v Bq_{i+1}, \mathrm{ad}_{q_i}\mathrm{ad}_v Bq_{i+2}\Big) \\
= \; & \frac{1}{2}\int K\Big(\mathrm{ad}_v Bq_{i+1}, \mathrm{ad}_{q_i}\mathrm{ad}_v Bq_{i+2}\Big) + \frac{1}{2}\int K\Big(\mathrm{ad}_v Bq_{i+1}, \mathrm{ad}_{q_i}\mathrm{ad}_v Bq_{i+2}\Big) \\
= \; & \frac{1}{2}\int K\Big(\mathrm{ad}_v Bq_{i+1}, \mathrm{ad}_{q_i}\mathrm{ad}_v Bq_{i+2}\Big) - \frac{1}{2}\int K\Big(\mathrm{ad}_{q_i}\mathrm{ad}_v Bq_{i+1}, \mathrm{ad}_v Bq_{i+2}\Big) \\
= \; & \frac{1}{2}\int K\Big(\mathrm{ad}_v Bq_{i+1}, -\mathrm{ad}_v\mathrm{ad}_{Bq_{i+2}}q_i - \mathrm{ad}_{Bq_{i+2}}\mathrm{ad}_{q_i}v\Big) \\
- \; & \frac{1}{2}\int K\Big(-\mathrm{ad}_v\mathrm{ad}_{Bq_{i+1}}q_i - \mathrm{ad}_{Bq_{i+1}}\mathrm{ad}_{q_i}v, \mathrm{ad}_v Bq_{i+2}\Big) \\
= \; & \frac{1}{2}\int K\Big(\mathrm{ad}_v\mathrm{ad}_{Bq_{i+1}}q_i, \mathrm{ad}_v Bq_{i+2}\Big) - \frac{1}{2}\int K\Big(\mathrm{ad}_v Bq_{i+1}, \mathrm{ad}_v\mathrm{ad}_{Bq_{i+2}}q_i\Big) \\
+ \; & \frac{1}{2}\int K\Big(\mathrm{ad}_{Bq_{i+1}}\mathrm{ad}_{q_i}v, \mathrm{ad}_v Bq_{i+2}\Big) - \frac{1}{2}\int K\Big(\mathrm{ad}_v Bq_{i+1}, \mathrm{ad}_{Bq_{i+2}}\mathrm{ad}_{q_i}v\Big)
\end{aligned}
$$

Now the expression becomes as follows using the invariance of the Killing.

$$
\begin{aligned}
&\int K\Big(\mathrm{ad}_v Bq_{i+1}, \mathrm{ad}_{q_i}\mathrm{ad}_v Bq_{i+2}\Big) \\
= \;& \frac{1}{2}\int K\Big(\mathrm{ad}_v\mathrm{ad}_v Bq_{i+1}, \mathrm{ad}_{Bq_{i+2}}q_i\Big) - \frac{1}{2}\int K\Big(\mathrm{ad}_v\mathrm{ad}_v\mathrm{ad}_{Bq_{i+1}}q_i, Bq_{i+2}\Big) \\
+ \;& \frac{1}{2}\int K\Big(\mathrm{ad}_{Bq_{i+2}}\mathrm{ad}_v Bq_{i+1}, \mathrm{ad}_{q_i}v\Big) - \frac{1}{2}\int K\Big(\mathrm{ad}_{Bq_{i+1}}\mathrm{ad}_v Bq_{i+2}, \mathrm{ad}_{q_i}v\Big) \\
= \;& -\frac{1}{2}\int K\Big(\mathrm{ad}_v\mathrm{ad}_v Bq_{i+1}, \mathrm{ad}_{q_i}Bq_{i+2}\Big) - \frac{1}{2}\int K\Big(\mathrm{ad}_{Bq_{i+1}}q_i, \mathrm{ad}_v\mathrm{ad}_v Bq_{i+2}\Big) \\
+ \;& \frac{1}{2}\int K\Big(\mathrm{ad}_{Bq_{i+2}}\mathrm{ad}_v Bq_{i+1}, \mathrm{ad}_{q_i}v\Big) - \frac{1}{2}\int K\Big(\mathrm{ad}_{Bq_{i+1}}\mathrm{ad}_v Bq_{i+2}, \mathrm{ad}_{q_i}v\Big)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\sum_{i\in\mathbb{Z}/3}\int K\Big(\mathrm{ad}_v Bq_{i+1}, \mathrm{ad}_{q_i}\mathrm{ad}_v Bq_{i+2}\Big) \\
= \;& \frac{1}{2}\sum_{i\in\mathbb{Z}/3}\int K\Big(\mathrm{ad}_v\mathrm{ad}_v Bq_{i+1}, \mathrm{ad}_{q_{i+2}}Bq_i - \mathrm{ad}_{q_i}Bq_{i+2}\Big) - K\Big(\mathrm{ad}_v\mathrm{ad}_{Bq_{i+1}}Bq_{i+2}, \mathrm{ad}_{q_i}v\Big) \\
= \;& \frac{1}{2}\sum_{i\in\mathbb{Z}/3}\int K\Big(\mathrm{ad}_v\mathrm{ad}_v Bq_{i+1}, \mathrm{ad}_{q_{i+2}}Bq_i - \mathrm{ad}_{q_i}Bq_{i+2}\Big) + K\Big(\mathrm{ad}_{Bq_{i+1}}Bq_{i+2}, \mathrm{ad}_v\mathrm{ad}_{q_i}v\Big)
\end{aligned}
$$

It follows from Lemma 6.2.5 that the following identity holds for the operator B:

$$
\int K\Big(\mathrm{ad}_{Bq_{i+1}}Bq_{i+2}, \cdot\Big) = \int K\Big(B\mathrm{ad}_{q_{i+1}}Bq_{i+2} - B\mathrm{ad}_{q_{i+2}}Bq_{i+1}, \cdot\Big).
$$

We find that

$$
\begin{aligned}
&\sum_{i\in\mathbb{Z}/3}\int K\Big(\mathrm{ad}_v Bq_{i+1}, \mathrm{ad}_{q_i}\mathrm{ad}_v Bq_{i+2}\Big) \\
= \;& \frac{1}{2}\sum_{i\in\mathbb{Z}/3}\int K\Big(\mathrm{ad}_v\mathrm{ad}_v Bq_{i+1}, \mathrm{ad}_{q_{i+2}}Bq_i - \mathrm{ad}_{q_i}Bq_{i+2}\Big) \\
- \;& \frac{1}{2}\sum_{i\in\mathbb{Z}/3}\int K\Big(B\mathrm{ad}_v\mathrm{ad}_{q_{i+1}}v, \mathrm{ad}_{q_{i+2}}Bq_i - \mathrm{ad}_{q_i}Bq_{i+2}\Big). \\
= \;& \frac{1}{2}\sum_{i\in\mathbb{Z}/3}\int K\Big(\mathrm{ad}_v\mathrm{ad}_v Bq_{i+1} - B\mathrm{ad}_v\mathrm{ad}_{q_{i+1}}v, \mathrm{ad}_{q_{i+2}}Bq_i - \mathrm{ad}_{q_i}Bq_{i+2}\Big) \\
= \;& \frac{1}{2}\sum_{i\in\mathbb{Z}/3}\int K\Big(\mathrm{ad}_v\mathrm{ad}_v Bq_{i+1} + B\mathrm{ad}_v\mathrm{ad}_v q_{i+1}, \mathrm{ad}_{q_{i+2}}Bq_i - \mathrm{ad}_{q_i}Bq_{i+2}\Big)
\end{aligned}
$$

■

Lemma B.0.12.

$$\begin{aligned}
&\int K\Big(D_xB\mathrm{ad}_{q_{i+2}}Bq_i - D_xB\mathrm{ad}_{q_i}Bq_{i+2}, \mathrm{ad}_{D_xv}Bq_{i+1} + D_xq_{i+1}\Big)\\
= &\ \frac{1}{2}\int K\Big(\mathrm{ad}_{q_{i+2}}Bq_i - \mathrm{ad}_{q_i}Bq_{i+2}, B\mathrm{ad}_v\mathrm{ad}_{D_xv}Bq_{i+1}\Big)\\
+ &\ \frac{1}{4}\int K\Big(\mathrm{ad}_{q_{i+2}}Bq_i - \mathrm{ad}_{q_i}Bq_{i+2}, B\mathrm{ad}_v\mathrm{ad}_vq_{i+1} + \mathrm{ad}_vq_{i+1}\Big)\\
+ &\ \frac{1}{4}\int K\Big(\mathrm{ad}_vBq_{i+2} + q_{i+2}, \mathrm{ad}_{q_{i+1}}q_i\Big).
\end{aligned}$$

Proof. Remember that $D_xB = \frac{1}{2} + \frac{1}{2}\mathrm{ad}_vB$.

$$\begin{aligned}
&\int K\Big(D_xB\mathrm{ad}_{q_{i+2}}Bq_i - D_xB\mathrm{ad}_{q_i}Bq_{i+2}, \mathrm{ad}_{D_xv}Bq_{i+1} + D_xq_{i+1}\Big)\\
= &\ \frac{1}{2}\int K\Big(\mathrm{ad}_vB\mathrm{ad}_{q_{i+2}}Bq_i - \mathrm{ad}_vB\mathrm{ad}_{q_i}Bq_{i+2}, \mathrm{ad}_{D_xv}Bq_{i+1} + D_xq_{i+1}\Big)\\
+ &\ \frac{1}{2}\int K\Big(\mathrm{ad}_{q_{i+2}}Bq_i - \mathrm{ad}_{q_i}Bq_{i+2}, \mathrm{ad}_{D_xv}Bq_{i+1} + D_xq_{i+1}\Big)
\end{aligned}$$

and it follows that

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}/3}\int K\Big(D_xB\mathrm{ad}_{q_{i+2}}Bq_i - D_xB\mathrm{ad}_{q_i}Bq_{i+2}, \mathrm{ad}_{D_xv}Bq_{i+1} + D_xq_{i+1}\Big)\\
= &\ \frac{1}{2}\sum_{i\in\mathbb{Z}/3}\int K\Big(\mathrm{ad}_vB\mathrm{ad}_{q_{i+2}}Bq_i - \mathrm{ad}_vB\mathrm{ad}_{q_i}Bq_{i+2}, \mathrm{ad}_{D_xv}Bq_{i+1} + D_xq_{i+1}\Big)\\
- &\ \frac{1}{2}\sum_{i\in\mathbb{Z}/3}\int K\Big(\mathrm{ad}_{D_xv}\mathrm{ad}_{q_{i+2}}Bq_i, Bq_{i+1}\Big) + \frac{1}{2}\sum_{i\in\mathbb{Z}/3}\int K\Big(Bq_{i+2}, \mathrm{ad}_{q_i}\mathrm{ad}_{D_xv}Bq_{i+1}\Big)\\
- &\ \frac{1}{2}\sum_{i\in\mathbb{Z}/3}\int K\Big(Bq_i, \mathrm{ad}_{q_{i+2}}D_xq_{i+1}\Big) + \frac{1}{2}\sum_{i\in\mathbb{Z}/3}\int K(Bq_{i+2}, \mathrm{ad}_{q_i}D_xq_{i+1})\\
= &\ \frac{1}{2}\sum_{i\in\mathbb{Z}/3}\int K\Big(\mathrm{ad}_vB\mathrm{ad}_{q_{i+2}}Bq_i - \mathrm{ad}_vB\mathrm{ad}_{q_i}Bq_{i+2}, \mathrm{ad}_{D_xv}Bq_{i+1} + D_xq_{i+1}\Big)\\
- &\ \frac{1}{2}\sum_{i\in\mathbb{Z}/3}\int K\Big(\mathrm{ad}_{D_xv}\mathrm{ad}_{q_i}Bq_{i+1}, Bq_{i+2}\Big) + \frac{1}{2}\sum_{i\in\mathbb{Z}/3}\int K\Big(Bq_{i+2}, \mathrm{ad}_{q_i}\mathrm{ad}_{D_xv}Bq_{i+1}\Big)\\
- &\ \frac{1}{2}\sum_{i\in\mathbb{Z}/3}\int K\Big(Bq_{i+2}, \mathrm{ad}_{q_{i+1}}D_xq_i\Big) + \frac{1}{2}\sum_{i\in\mathbb{Z}/3}\int K(Bq_{i+2}, \mathrm{ad}_{q_i}D_xq_{i+1})\\
= &\ \frac{1}{2}\sum_{i\in\mathbb{Z}/3}\int K\Big(\mathrm{ad}_vB\mathrm{ad}_{q_{i+2}}Bq_i - \mathrm{ad}_vB\mathrm{ad}_{q_i}Bq_{i+2}, \mathrm{ad}_{D_xv}Bq_{i+1} + D_xq_{i+1}\Big)\\
- &\ \frac{1}{2}\sum_{i\in\mathbb{Z}/3}\int K\Big(\mathrm{ad}_{Bq_{i+1}}\mathrm{ad}_{q_i}D_xv, Bq_{i+2}\Big) - \frac{1}{2}\sum_{i\in\mathbb{Z}/3}\int K\Big(Bq_{i+2}, D_x\mathrm{ad}_{q_{i+1}}q_i\Big)
\end{aligned}$$

where the last two terms where obtained by applying the Jacobi identity and the derivation property. We now have, using again Lemma 6.2.5,

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}/3}\int K\Big(D_xB\mathrm{ad}_{q_{i+2}}Bq_i - D_xB\mathrm{ad}_{q_i}Bq_{i+2}, \mathrm{ad}_{D_xv}Bq_{i+1} + D_xq_{i+1}\Big)\\
=\;& \frac{1}{2}\sum_{i\in\mathbb{Z}/3}\int K\Big(\mathrm{ad}_vB\mathrm{ad}_{q_{i+2}}Bq_i - \mathrm{ad}_vB\mathrm{ad}_{q_i}Bq_{i+2}, \mathrm{ad}_{D_xv}Bq_{i+1} + D_xq_{i+1}\Big)\\
+\;& \frac{1}{2}\sum_{i\in\mathbb{Z}/3}\int K\Big(\mathrm{ad}_{Bq_{i+1}}Bq_{i+2}, \mathrm{ad}_{q_i}D_xv\Big) - \frac{1}{2}\sum_{i\in\mathbb{Z}/3}\int K\Big(Bq_{i+2}, D_x\mathrm{ad}_{q_{i+1}}q_i\Big)\\
=\;& \frac{1}{2}\sum_{i\in\mathbb{Z}/3}\int K\Big(\mathrm{ad}_{q_{i+1}}Bq_{i+2} - \mathrm{ad}_{q_{i+2}}Bq_{i+1}\\
&, B\mathrm{ad}_v\mathrm{ad}_{D_xv}Bq_i + B\mathrm{ad}_vD_xq_i - B\mathrm{ad}_{q_i}D_xv\Big)\\
+\;& \frac{1}{4}\sum_{i\in\mathbb{Z}/3}\int K\Big(\mathrm{ad}_vBq_{i+2} + q_{i+2}, \mathrm{ad}_{q_{i+1}}q_i\Big)\\
=\;& \frac{1}{2}\sum_{i\in\mathbb{Z}/3}\int K\Big(\mathrm{ad}_{q_{i+1}}Bq_{i+2} - \mathrm{ad}_{q_{i+2}}Bq_{i+1}, B\mathrm{ad}_v\mathrm{ad}_{D_xv}Bq_i + BD_x\mathrm{ad}_vq_i\Big)\\
+\;& \frac{1}{4}\sum_{i\in\mathbb{Z}/3}\int K\Big(\mathrm{ad}_vBq_{i+2} + q_{i+2}, \mathrm{ad}_{q_{i+1}}q_i\Big)\\
=\;& \frac{1}{2}\sum_{i\in\mathbb{Z}/3}\int K\Big(\mathrm{ad}_{q_{i+1}}Bq_{i+2} - \mathrm{ad}_{q_{i+2}}Bq_{i+1}, B\mathrm{ad}_v\mathrm{ad}_{D_xv}Bq_i\Big)\\
+\;& \frac{1}{4}\sum_{i\in\mathbb{Z}/3}\int K\Big(\mathrm{ad}_{q_{i+1}}Bq_{i+2} - \mathrm{ad}_{q_{i+2}}Bq_{i+1}, \mathrm{ad}_vq_i + B\mathrm{ad}_v\mathrm{ad}_vq_i\Big)\\
+\;& \frac{1}{4}\sum_{i\in\mathbb{Z}/3}\int K\Big(\mathrm{ad}_vBq_{i+2} + q_{i+2}, \mathrm{ad}_{q_{i+1}}q_i\Big)\\
=\;& \frac{1}{2}\sum_{i\in\mathbb{Z}/3}\int K\Big(\mathrm{ad}_{q_{i+2}}Bq_i - \mathrm{ad}_{q_i}Bq_{i+2}, B\mathrm{ad}_v\mathrm{ad}_{D_xv}Bq_{i+1}\Big)\\
+\;& \frac{1}{4}\sum_{i\in\mathbb{Z}/3}\int K\Big(\mathrm{ad}_{q_{i+2}}Bq_i - \mathrm{ad}_{q_i}Bq_{i+2}, \mathrm{ad}_vq_{i+1} + B\mathrm{ad}_v\mathrm{ad}_vq_{i+1}\Big)\\
+\;& \frac{1}{4}\sum_{i\in\mathbb{Z}/3}\int K\Big(\mathrm{ad}_vBq_{i+2} + q_{i+2}, \mathrm{ad}_{q_{i+1}}q_i\Big)
\end{aligned}$$

where we used the relation $BD_x = \frac{1}{2} + \frac{1}{2}B\mathrm{ad}_v$. ■

Lemma B.0.13.

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}/3}\int K\Big(q_{i+1},\mathrm{ad}_{q_i}\mathrm{ad}_v Bq_{i+2}\Big)\\
= & -\sum_{i\in\mathbb{Z}/3}\int K\Big(\mathrm{ad}_v q_{i+1},\mathrm{ad}_{q_i}Bq_{i+2}-\mathrm{ad}_{q_{i+2}}Bq_i\Big).
\end{aligned}$$

Proof.

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}/3}\int K\Big(q_{i+1},\mathrm{ad}_{q_i}\mathrm{ad}_v Bq_{i+2}\Big)\\
= & -\sum_{i\in\mathbb{Z}/3}\int K\Big(q_{i+1},\mathrm{ad}_v\mathrm{ad}_{Bq_{i+2}}q_i\Big)-\sum_{i\in\mathbb{Z}/3}\int K\Big(q_{i+1},\mathrm{ad}_{B_{i+2}}\mathrm{ad}_{q_i}v\Big)\\
= & +\sum_{i\in\mathbb{Z}/3}\int K\Big(\mathrm{ad}_v q_{i+1},\mathrm{ad}_{Bq_{i+2}}q_i\Big)+\sum_{i\in\mathbb{Z}/3}\int K\Big(\mathrm{ad}_{Bq_{i+2}}q_{i+1},\mathrm{ad}_{q_i}v\Big)\\
= & -\sum_{i\in\mathbb{Z}/3}\int K\Big(\mathrm{ad}_v q_{i+1},\mathrm{ad}_{q_i}Bq_{i+2}\Big)-\sum_{i\in\mathbb{Z}/3}\int K\Big(\mathrm{ad}_{q_{i+2}}Bq_i,\mathrm{ad}_{q_{i+1}}v\Big)\\
= & -\sum_{i\in\mathbb{Z}/3}\int K\Big(\mathrm{ad}_v q_{i+1},\mathrm{ad}_{q_i}Bq_{i+2}-\mathrm{ad}_{q_{i+2}}Bq_i\Big).
\end{aligned}$$

■