Integrable Systems and Symplectic Geometry

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Appendix C

Computation of Killing form

Here we give a direct proof of Lemma 1.4.1.

Proof. Let X_1, X_2 be the subsets of $\mathfrak{sp}_n$ defined by

$$X_1 = \{E_{ss}^{(q)} \mid s = 1, \dots, n, \quad q = i, j, k\},$$

$$X_2 = \{F_{rs}^{(1)}, F_{rs}^{(i)}, F_{rs}^{(j)}, F_{rs}^{(k)} \mid r < s \quad \text{with} \quad r = 1, \dots, n-1, \quad s = 2, \dots, n\},$$

respectively, in which, for instance, $E_{ss}^{(i)}$ is a matrix with entry $(E_{ss}^{(i)})_{ss} = i$ and zero elsewhere , $F_{rs}^{(1)}$ with entry $(F_{rs}^{(1)})_{rs} = 1$ and $(F_{rs}^{(1)})_{sr} = -1$ and zero elsewhere, matrix $F_{rs}^{(i)}$ with entry $(F_{rs}^{(i)})_{rs} = i = (F_{rs}^{(i)})_{sr}$. and zero elsewhere. Then $X_1 \cup X_2$ is a basis for $\mathfrak{sp}(n, \mathbb{H})$.

Let us denote linear map $ad_A \circ ad_B$ shortly by T and take element E from the first set X_1 in which $E_{ss} \neq 0$. Then we can compute the matrices below.

$$(BE)_{rt} = \begin{cases} 0 \quad \text{if} \quad t \neq s \\ B_{rs}E_{ss} \quad \text{if} \quad t = s \end{cases}, \quad (EB)_{rt} = \begin{cases} 0 \quad \text{if} \quad r \neq s \\ E_{ss}B_{st} \quad \text{if} \quad r = s \end{cases}.$$

Hence

$$[B, E]_{rt} = \begin{cases} 0 \quad \text{if} \quad r, t \neq s \\ -E_{ss}B_{st} \quad \text{if} \quad r = s, t \neq s \\ B_{rs}E_{ss} \quad \text{if} \quad r \neq s, t = s \\ B_{ss}E_{ss} - E_{ss}B_{ss} \quad \text{if} \quad r = s, t = s \end{cases}.$$

Now we need to compute $(TE)_{ss}$. To do so we compute following entry of matrices $A[B, E]$ and $[B, E]A$:

$$(A[B, E])_{ss} = A_{ss}(B_{ss}E_{ss} - E_{ss}B_{ss}) + \sum_{k=1, k \neq s}^{n} A_{sk}B_{ks}E_{ss},$$

$$([B, E]A)_{ss} = (B_{ss}E_{ss} - E_{ss}B_{ss})A_{ss} + \sum_{k=1, k \neq s}^{n} (-E_{ss}B_{sk})A_{ks}.$$

Thus we have

$$\begin{aligned}(TE)_{ss} &= [A,[B,E]]_{ss} \\ &= (A_{ss}B_{ss}E_{ss}+E_{ss}B_{ss}A_{ss})-(A_{ss}E_{ss}B_{ss}+B_{ss}E_{ss}A_{ss}) \\ &\quad+\sum_{k=1,k\neq s}^{n}(A_{sk}B_{ks}E_{ss}+E_{ss}B_{sk}A_{ks}).\end{aligned}$$

Now we take element F from the second subset X_2 of basis in which $F_{pq}\neq 0$, for $p<q$. Then

$$(BF)_{rt}=B_{rp}F_{pt}+B_{rq}F_{qt}=\begin{cases}0 \quad \text{if} \quad t\neq p,q\\ B_{rq}F_{qp} \quad \text{if} \quad t=p\\ B_{rp}F_{pq} \quad \text{if} \quad t=q,\end{cases}$$

and

$$(FB)_{rt}=F_{rp}B_{pt}+F_{rq}B_{qt}=\begin{cases}0 \quad \text{if} \quad r\neq p,q\\ F_{pq}B_{qt} \quad \text{if} \quad r=p\\ F_{qp}B_{pt} \quad \text{if} \quad r=q,\end{cases}$$

Hence

$$[B,F]_{rt}=\begin{cases}0 \quad \text{if} \quad r\neq p,q\\ -F_{pq}B_{qt} \quad \text{if} \quad r=p,t\neq p,q\\ -F_{qp}B_{pt} \quad \text{if} \quad r=q,t\neq p,q\\ B_{rq}F_{qp} \quad \text{if} \quad t=p,r\neq p,q\\ B_{rp}F_{pq} \quad \text{if} \quad t=q,t\neq p,q\\ B_{pq}F_{qp}-F_{pq}B_{qp} \quad \text{if} \quad r=p,t=p\\ B_{pp}F_{pq}-F_{pq}B_{qq} \quad \text{if} \quad r=p,t=q\\ B_{qq}F_{qp}-F_{qp}B_{pp} \quad \text{if} \quad r=q,t=p\\ B_{qp}F_{pq}-F_{qp}B_{pq} \quad \text{if} \quad r=q,t=q\end{cases}$$

Thus we have that

$$\begin{aligned}(A[B,F])_{pq} &= A_{pp}(B_{pp}F_{pq}-F_{pq}B_{qq})+A_{pq}(B_{qp}F_{pq}-F_{qp}B_{pq}) \\ &\quad+\sum_{k=1,k\neq p,q}^{n}A_{pk}(B_{kp}F_{pq}),\end{aligned}$$

and

$$\begin{aligned}([B,F]A)_{pq} &= (B_{pq}F_{qp}-F_{pq}B_{qp})A_{pq}+(B_{pp}F_{pq}-F_{pq}B_{qq})A_{qq} \\ &\quad+\sum_{k=1,k\neq p,q}^{n}(-F_{pq}B_{qk})A_{kq}.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}(TF)_{pq} \quad = \quad & (A_{pp}B_{pp} + A_{pq}B_{qp})F_{pq} + F_{pq}(B_{qp}A_{pq} + B_{qq}A_{qq}) \\ & - (A_{pp}F_{pq}B_{qq} + B_{pp}F_{pq}A_{qq} + A_{pq}F_{qp}B_{pq} + B_{pq}F_{qp}A_{pq}) \\ & + \sum_{k=1, k\neq p,q} (A_{pk}B_{kp}F_{pq} + F_{pq}B_{qk}A_{kq}).\end{aligned}$$

Then $K(A,B)$ can be obtained as follows.

$$\begin{aligned} & K(A,B) = \sum_{s=1}^{n} (TE_{ss}^{(i)})^i + (TE_{ss}^{(j)})^j + (TE_{ss}^{(k)})^k \\ & + \sum_{p<q} (TF_{pq}^{(1)})^1 + (TF_{pq}^{(i)})^i + (TF_{pq}^{(j)})^j + (TF_{pq}^{(k)})^k \\ = \quad & \sum_{s=1}^{n} [-8 < A_{ss}, B_{ss} >_r -6 \sum_{k\neq s, k=1}^{n} < A_{sk}, B_{sk} >_r] \\ & + \sum_{p<q} [-4 < A_{pp}, B_{pp} >_r -4 < A_{qq}, B_{qq} >_r -12 < A_{pq}, B_{pq} >_r \\ & - 4 (\sum_{k\neq p,q} < A_{pk}, B_{pk} >_r + < A_{qk}, B_{qk} >_r)] \\ = \quad & -4(n+1) \sum_{s=1}^{n} < A_{ss}, B_{ss} >_r -8(n+1) \sum_{p<q} < A_{pq}, B_{pq} >_r, \end{aligned}$$

where the following identities have been used:

$$\sum_{p<q} [< A_{pp}, B_{pp} >_r + < A_{qq}, B_{qq} >_r = (n-1) \sum_{s=1}^{n} < A_{ss}, B_{ss} >_r,$$

$$\sum_{s=1}^{n} \sum_{k\neq s,k=1}^{n} < A_{sk}, B_{sk} >_r = 2 \sum_{p<q} < A_{pq}, B_{pq} >_r,$$

$$\sum_{p<q} \sum_{k\neq p,q} (< A_{pk}, B_{pk} >_r + < A_{qk}, B_{qk} >_r) = 2(n-1) \sum_{p<q} < A_{pq}, B_{pq} >_r .$$

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