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Appendix C

Computation of Killing form

Here we give a direct proof of Lemma 1.4.1.

Proof. Let X_1, X_2 be the subsets of \mathfrak{sp}_n defined by

$$X_1 = \{E_{ss}^{(q)} \mid s = 1, \dots, n, \quad q = i, j, k\},$$

$$X_2 = \{F_{rs}^{(1)}, F_{rs}^{(i)}, F_{rs}^{(j)}, F_{rs}^{(k)} \mid r < s \text{ with } r = 1, \dots, n-1, \quad s = 2, \dots, n\},$$

respectively, in which, for instance, $E_{ss}^{(i)}$ is a matrix with entry $(E_{ss}^{(i)})_{ss} = i$ and zero elsewhere, $F_{rs}^{(1)}$ with entry $(F_{rs}^{(1)})_{rs} = 1$ and $(F_{rs}^{(1)})_{sr} = -1$ and zero elsewhere, matrix $F_{rs}^{(i)}$ with entry $(F_{rs}^{(i)})_{rs} = i = (F_{rs}^{(i)})_{sr}$. and zero elsewhere. Then $X_1 \cup X_2$ is a basis for $\mathfrak{sp}(n, \mathbb{H})$.

Let us denote linear map $ad_A \circ ad_B$ shortly by T and take element E from the first set X_1 in which $E_{ss} \neq 0$. Then we can compute the matrices below.

$$(BE)_{rt} = \begin{cases} 0 & \text{if } t \neq s \\ B_{rs}E_{ss} & \text{if } t = s \end{cases}, \quad (EB)_{rt} = \begin{cases} 0 & \text{if } r \neq s \\ E_{ss}B_{st} & \text{if } r = s \end{cases}.$$

Hence

$$[B, E]_{rt} = \begin{cases} 0 & \text{if } r, t \neq s \\ -E_{ss}B_{st} & \text{if } r = s, t \neq s \\ B_{rs}E_{ss} & \text{if } r \neq s, t = s \\ B_{ss}E_{ss} - E_{ss}B_{ss} & \text{if } r = s, t = s \end{cases}.$$

Now we need to compute $(TE)_{ss}$. To do so we compute following entry of matrices $A[B, E]$ and $[B, E]A$:

$$(A[B, E])_{ss} = A_{ss}(B_{ss}E_{ss} - E_{ss}B_{ss}) + \sum_{k=1, k \neq s}^n A_{sk}B_{ks}E_{ss},$$

$$([B, E]A)_{ss} = (B_{ss}E_{ss} - E_{ss}B_{ss})A_{ss} + \sum_{k=1, k \neq s}^n (-E_{ss}B_{sk})A_{ks}.$$

Thus we have

$$\begin{aligned}
(TE)_{ss} &= [A, [B, E]]_{ss} \\
&= (A_{ss}B_{ss}E_{ss} + E_{ss}B_{ss}A_{ss}) - (A_{ss}E_{ss}B_{ss} + B_{ss}E_{ss}A_{ss}) \\
&\quad + \sum_{k=1, k \neq s}^n (A_{sk}B_{ks}E_{ss} + E_{ss}B_{sk}A_{ks}).
\end{aligned}$$

Now we take element F from the second subset X_2 of basis in which $F_{pq} \neq 0$, for $p < q$. Then

$$(BF)_{rt} = B_{rp}F_{pt} + B_{rq}F_{qt} = \begin{cases} 0 & \text{if } t \neq p, q \\ B_{rq}F_{qp} & \text{if } t = p \\ B_{rp}F_{pq} & \text{if } t = q, \end{cases}$$

and

$$(FB)_{rt} = F_{rp}B_{pt} + F_{rq}B_{qt} = \begin{cases} 0 & \text{if } r \neq p, q \\ F_{pq}B_{qt} & \text{if } r = p \\ F_{qp}B_{pt} & \text{if } r = q, \end{cases}$$

Hence

$$[B, F]_{rt} = \begin{cases} 0 & \text{if } r \neq p, q \\ -F_{pq}B_{qt} & \text{if } r = p, t \neq p, q \\ -F_{qp}B_{pt} & \text{if } r = q, t \neq p, q \\ B_{rq}F_{qp} & \text{if } t = p, r \neq p, q \\ B_{rp}F_{pq} & \text{if } t = q, t \neq p, q \\ B_{pq}F_{qp} - F_{pq}B_{qp} & \text{if } r = p, t = p \\ B_{pp}F_{pq} - F_{pq}B_{qq} & \text{if } r = p, t = q \\ B_{qq}F_{qp} - F_{qp}B_{pp} & \text{if } r = q, t = p \\ B_{qp}F_{pq} - F_{qp}B_{pq} & \text{if } r = q, t = q \end{cases}$$

Thus we have that

$$\begin{aligned}
(A[B, F])_{pq} &= A_{pp}(B_{pp}F_{pq} - F_{pq}B_{qq}) + A_{pq}(B_{qp}F_{pq} - F_{qp}B_{pq}) \\
&\quad + \sum_{k=1, k \neq p, q}^n A_{pk}(B_{kp}F_{pq}),
\end{aligned}$$

and

$$\begin{aligned}
([B, F]A)_{pq} &= (B_{pq}F_{qp} - F_{pq}B_{qp})A_{pq} + (B_{pp}F_{pq} - F_{pq}B_{qq})A_{qq} \\
&\quad + \sum_{k=1, k \neq p, q}^n (-F_{pq}B_{qk})A_{kq}.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
(TF)_{pq} &= (A_{pp}B_{pp} + A_{pq}B_{qp})F_{pq} + F_{pq}(B_{qp}A_{pq} + B_{qq}A_{qq}) \\
&\quad - (A_{pp}F_{pq}B_{qq} + B_{pp}F_{pq}A_{qq} + A_{pq}F_{qp}B_{pq} + B_{pq}F_{qp}A_{pq}) \\
&\quad + \sum_{k=1, k \neq p, q} (A_{pk}B_{kp}F_{pq} + F_{pq}B_{qk}A_{kq}).
\end{aligned}$$

Then $K(A, B)$ can be obtained as follows.

$$\begin{aligned}
K(A, B) &= \sum_{s=1}^n (TE_{ss}^{(i)})^i + (TE_{ss}^{(j)})^j + (TE_{ss}^{(k)})^k \\
&\quad + \sum_{p < q} (TF_{pq}^{(1)})^1 + (TF_{pq}^{(i)})^i + (TF_{pq}^{(j)})^j + (TF_{pq}^{(k)})^k \\
&= \sum_{s=1}^n [-8 \langle A_{ss}, B_{ss} \rangle_r - 6 \sum_{k \neq s, k=1}^n \langle A_{sk}, B_{sk} \rangle_r] \\
&\quad + \sum_{p < q} [-4 \langle A_{pp}, B_{pp} \rangle_r - 4 \langle A_{qq}, B_{qq} \rangle_r - 12 \langle A_{pq}, B_{pq} \rangle_r \\
&\quad - 4(\sum_{k \neq p, q} \langle A_{pk}, B_{pk} \rangle_r + \langle A_{qk}, B_{qk} \rangle_r)] \\
&= -4(n+1) \sum_{s=1}^n \langle A_{ss}, B_{ss} \rangle_r - 8(n+1) \sum_{p < q} \langle A_{pq}, B_{pq} \rangle_r,
\end{aligned}$$

where the following identities have been used:

$$\begin{aligned}
\sum_{p < q} [\langle A_{pp}, B_{pp} \rangle_r + \langle A_{qq}, B_{qq} \rangle_r] &= (n-1) \sum_{s=1}^n \langle A_{ss}, B_{ss} \rangle_r, \\
\sum_{s=1}^n \sum_{k \neq s, k=1}^n \langle A_{sk}, B_{sk} \rangle_r &= 2 \sum_{p < q} \langle A_{pq}, B_{pq} \rangle_r, \\
\sum_{p < q} \sum_{k \neq p, q} (\langle A_{pk}, B_{pk} \rangle_r + \langle A_{qk}, B_{qk} \rangle_r) &= 2(n-1) \sum_{p < q} \langle A_{pq}, B_{pq} \rangle_r.
\end{aligned}$$

■

