Kappa-Ohio completeness and related problems

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2009

Summary

The property of Ohio completeness was introduced by Arhangel′skiĭ in [2] to study properties of remainders in compactifications. This property is very interesting in itself and in [7] it was generalized to the κ-Ohio completeness property. In this dissertation we study κ-Ohio completeness in a systematic way.

The first chapter contains preliminary results from general topology and can be skipped by an experienced topologist.

We start the second chapter by giving examples of classes of spaces that are κ-Ohio complete, for a fixed cardinal κ; it turns out that spaces with either Lindelöf or Čech-number at most κ, spaces with a G_κ-diagonal, and p_κ-spaces are κ-Ohio complete. Then we study subspaces of κ-Ohio complete spaces. Since the κ-Ohio completeness property is not hereditary in general, we focus on the question whether κ-Ohio completeness is closed-hereditary. We do not know the answer to this question, but we prove that closed and C^*-embedded subspaces of κ-Ohio complete spaces are again κ-Ohio complete. Hence, in normal spaces, κ-Ohio completeness is closed-hereditary. Furthermore, for uncountable cardinals, we prove a characterization of closed subspaces of κ-Ohio complete spaces. In the last part of the second chapter we deal with sum theorems for κ-Ohio completeness. We prove that κ-Ohio completeness is preserved by taking both κ-locally finite closed sums and point-finite open sums. So in particular κ-Ohio completeness is preserved under taking topological sums. Furthermore we prove that, under the assumption $\mathfrak{d} = \omega_1$, the countable union of ω_1-Ohio complete spaces is again ω_1-Ohio complete.

The third chapter is devoted to products of κ-Ohio complete spaces. It is unknown whether κ-Ohio completeness is preserved by taking the

product with a compact space; it turns out that if it were so, then κ-Ohio completeness would be closed-hereditary. For infinite products, we have negative results; indeed, we prove that, for every cardinal κ less than the first weakly inaccessible cardinal, the space ω^{κ^+} is not κ-Ohio complete. Furthermore, we prove that ω^{ω_1} cannot be embedded as a closed subspace in any Ohio complete space.

In Chapter 4 we collect examples and counterexamples that provide answers to some questions from the second chapter. We start by providing several examples of homogeneous first-countable spaces that are not Ohio complete; this answers Question 2.2.13 for the case $\kappa = \omega$. We then present a non κ-Ohio complete space that is the union of a locally countable family of closed and κ-Ohio complete subspaces; this answers the question asked right after Corollary 2.5.6. After that we prove that if κ is a cardinal number strictly less than $\mathfrak{d}$, then the countable union of open and Ohio complete subspaces need not be κ-Ohio complete; and that if κ is regular, the union of κ-many open and κ-Ohio complete subspaces need not be κ-Ohio complete. Finally we construct examples of homogeneous spaces that are of point-countable type but not of countable type. These examples are related to a theorem of Pasynkov from [39] where it is proved that, in topological groups, being of point-countable type is equivalent to being of countable type. Thus our examples show that the result from [39] cannot be improved.

The topic of the last chapter is the Čech-number of Σ-products. The inspiration for such a subject comes from a result about ω_1-Ohio completeness. From Lemma 4.4.1 it follows that $\check{C}(\Sigma 2^{\omega_1})$ is strictly greater than ω_1. So natural questions are: what can we say about this number? Is it a well-known cardinal number? We do not provide a complete answer, but we characterize $\check{C}(\Sigma 2^{\omega_1})$ in terms of dominating families in a suitably chosen poset. We also show that it may be strictly bigger and strictly less than $\mathfrak{d}$.