Asymptotics in Deconvolution Models

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Five

Limit Distribution

In this chapter a conjecture about the pointwise asymptotic limit distribution of $\hat{F}_n$ is discussed. In order to show that this pointwise limit equals the last time that a two sided Brownian motion minus a parabola reaches its maximum, we use in our present approach many results from previous chapters, in particular the decomposition of the Fenchel process and the cube-root-n behavior of $\hat{F}_n$ and $\hat{h}_n$ (Section 5.1). Unfortunately we could not finish the argument at this stage since we still need to derive that specific functionals of $\hat{F}_n$ are smooth in the sense that they converge at rate $n^{-1/2}$ to their true values. A sketch of how we want to derive this rate is shown in Section 5.2 together with a discussion of the essential difficulties.

5.1 Pointwise Limit Distribution

Groeneboom and Wellner (1992) state a conjecture about the pointwise asymptotic behavior of the MLE $\hat{F}_n$ in deconvolution models with decreasing noise density in which they claim that the (properly scaled) pointwise distributional limit of $\hat{F}_n$ equals $2U$, i.e.

$$n^{1/3} c_{F_0} \left(\hat{F}_n(x_0) - F_0(x_0) \right) \quad \stackrel{\mathrm{d}}{\to} \quad 2U \tag{5.1.1}$$

where c_{F_0} denotes some constant depending on F_0 and where U is the last time that a two-sided Brownian motion with parabolic drift has its maximum. The distribution of U is called Chernoff's distribution.
Groeneboom and Wellner (1992) also provide some further insight in the validity of the conjecture by their study of the 'one-step-estimator', the 'estimator' one gets after performing one step of an iterative algorithm, starting with the underlying distribution function F_0. This 'one-step-estimator' converges to $2U$ in the sense of (5.1.1) so that, under the hypothesis that $\hat{F}_n$ and this 'estimator' are first order asymptotically equivalent in a pointwise sense, (5.1.1) holds. Furthermore, in the case of g being the standard exponential density, Jongbloed (1998a) derived (5.1.1). The proof relies heavily on the special structure of the exponential density g, which allows an explicit construction of the MLE in this particular example, and can therefore not be used for the more general cases we consider. It also uses the fact, that the Isotonic Inverse Estimator (IIE), an alternative, explicitly defined nonparametric estimator for F_0 in the deconvolution model, converges to the distribution conjectured for the MLE (see van Es et al., 1998)).

We want to study the limit behavior of the MLE via the Fenchel conditions, as there is no closed expression for the estimator itself. The same idea has been used to derive the asymptotic behavior of maximum likelihood estimators in other inverse problems or estimation problems under shape constraints. See for example Groeneboom et al. (2001), a paper that deals with the asymptotic behavior of convex density estimators. Also in the study of the Current Status model with competing risks, which was done recently in Groeneboom et al. (2008c), one can find a similar approach. However, there is no unified methodology yet for obtaining the asymptotic limit distribution from the optimality conditions.
Using this approach in the present situation, we face the difficult problem that specific functionals of the MLE have to be 'smooth' as they require a rate $n^{-1/2}$ for converging to their true values while $\hat{F}_n$ itself converges only at rate $n^{-1/3}$ (see Chapter 4). Establishing smoothness turns out to be a hard problem we did not succeed yet to complete but which is part of ongoing research. In that sense we cannot give a proof of (5.1.1) but still want to point out the general line of arguments we intend to follow.
Most of the results concerning consistency and rates of convergence derived in earlier parts will be used. Therefore, the deconvolution model and x_0 in this chapter is chosen according to the following assumption although we do believe that (5.1.1) also holds under less restrictive conditions like, for instance, non compact support of g.

Assumption 5.1.1.

- *F_0 is continuous with $S_0 < \infty$.*
- *The density g is bounded, decreasing and smooth on $(0, \infty)$ as in (2.1.3). It has a compact support, i.e. $S_g < \infty$, and possesses a bounded Lipschitz continuous derivative on $(0, S_g)$.*
- *In a neighborhood of $x_0 \in (0, S_0)$, the density f_0 is continuous with $f_0(x_0) > 0$.*

Remark.
Assumption 5.1.1 implies that there exist constants $\lambda > 0$ and $c_\lambda > 0$ such that $h_0(z) \geq c_\lambda$ for all $x \in [\,x_0 - \lambda, x_0 + S_g + \lambda\,]$ with x_0 chosen as in Assumption 5.1.1. Moreover, $\hat{h}_n(z) \geq c_\lambda$ on the same interval with probability tending to one; see also Lemma 4.2.1 and the Remark thereafter. Hence, for sufficiently large n we obtain that $\hat{h}_n$ is bounded from below by c_λ on $[\,x_0 - n^{-1/3}c, x_0 + n^{-1/3}c\,]$ with arbitrarily high probability for some $c > 0$.

As already mentioned, the square-root-n behavior of specific functionals of $\hat{F}_n$ is crucial in the proof of (5.1.1), but not yet derived. We therefore state the needed behavior of the functionals as a conjecture (below) and show how (5.1.1) can be derived under that conjecture. The following section then contains a discussion of a possible line of a proof for Conjecture 5.1.2.

Conjecture 5.1.2.
Let x_0 be as in Assumption 5.1.1 and define for $t \in (0, S_F)$ and $F \in \mathcal{F}_{[\,0,\infty)}$

$$\begin{aligned} K_t(F) &= \int_{[\,0,t)} (g(t-x) - g(0))\; dF(x) \;\; \textit{and} \\ \tilde{K}_{t,s}(F) &= \int_{[\,t,t+S_g\,]} \frac{g(z-t) - g(z-s)}{\hat{h}_n(z)}\, dH_F(z). \end{aligned}$$

Then one can find a small $\lambda > 0$ such that

$$\begin{aligned} \sup_{t \in [\,x_0-\lambda, x_0+\lambda\,]} \Big(K_t(\hat{F}_n) - K_t(F_0) \Big) &= O_p(n^{-1/2}) \;\; \textit{and} \qquad (5.1.2) \\ \sup_{(t,s) \in [\,x_0-\lambda, x_0+\lambda\,]^2} \Big(\tilde{K}_{s,t}(\hat{F}_n) - \tilde{K}_{s,t}(F_0) \Big) &= O_p(n^{-1/2}). \end{aligned}$$

Theorem 5.1.3 (Pointwise Asymptotic Distribution of $\hat{F}_n$).
Under Assumption 5.1.1 and Conjecture 5.1.2,

$$n^{1/3} \left(\frac{2g(0)^2}{f_0(x_0) h_0(x_0)} \right)^{1/3} \Big(\hat{F}_n(x_0) - F_0(x_0) \Big) \xrightarrow{d} 2U \qquad (5.1.3)$$

where $\xrightarrow{d}$ denotes convergence in distribution and where U is the last time that a standard two-sided Wiener process originating from zero minus a parabola reaches its maximum.

The distribution of the random variable U is known as 'Chernoff's distribution' in honor of Chernoff who first introduced it in 1964. Prakasa Rao (1969) showed that the limit process $2U$ can also be described in terms of the derivative of the concave majorant at zero of a Brownian motion with negative quadratic drift (see Figure 5.1). Its density can be calculated by solving a differential equation involving Airy functions (see Groeneboom (1989) and Groeneboom and Wellner (1999)). Figure 5.2 shows the density of U. The difference to the normal density having the same variance as U is difficult to see if both densities are plotted in one picture. Figures 5.3 and 5.4 illustrate the discrepancy of the two densities by their difference and their ratio, respectively. The latter shows in particular that the tails of Chernoff's distribution are thinner. This corresponds to the fact that $f_{N(0,0.2635)} \sim c_1 \exp(-c_2 t^2)$ and $f_U \sim c_3 \exp(-c_4 t^3)$ for constants $c_i > 0, i = 1, \ldots, 4$, as $t \to \infty$, where the behavior of f_U is given in Groeneboom and Wellner (1999).

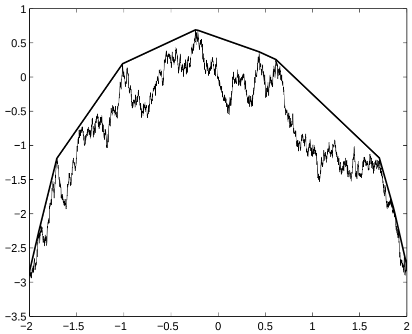

FIGURE 5.1: Concave Majorant of BM-t^2.

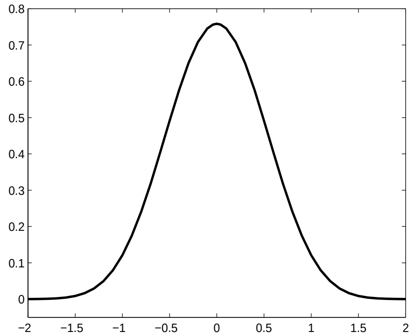

FIGURE 5.2: Density of U.

Under Conjecture 5.1.2, the limit behavior of $\hat{F}_n$ can be deduced from the Fenchel optimality conditions via their decomposition derived in (4.2.3). For x_0 taken as in Assumption 5.1.1, we define a localized version $\hat{C}_n^{\text{loc}} : \mathbb{R} \to \mathbb{R}$ of the Fenchel process $\hat{C}_n$ by

$$\hat{C}_n^{\text{loc}}(x) \quad = \quad n^{2/3} \left\{ \hat{C}_n(x_0 + n^{-1/3}x) - \hat{C}_n(x_0) \right\}.$$

Then Theorem 2.2.1 (Fenchel conditions) immediately implies that $\hat{C}_n^{\text{loc}}(x) \leq -A_n$ for all $x \in \mathbb{R}$ and for $A_n = n^{2/3}(\hat{C}_n(x_0) - 1) = n^{2/3}(\hat{C}_n(x_0) - \hat{C}_n(\tau^+))$ where $\tau^+ \in \mathcal{T}_n$ (see Definition 2.1.3) denotes the first point of jump of $\hat{F}_n$ to the right of x_0. Moreover, $\hat{C}_n^{\text{loc}}(x) = -A_n$ whenever $x_0 + n^{-1/3}x \in \mathcal{T}_n$. Hence, by letting $s = x_0$ and $t = x_0 + n^{-1/3}x$

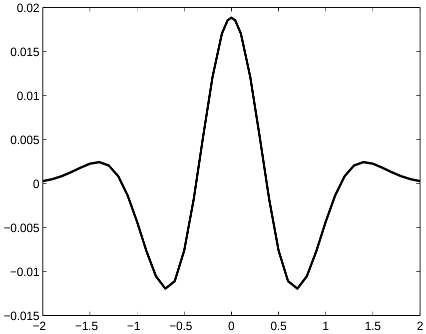

FIGURE 5.3: Difference $f_{N(0,0.2635)} - f_U$.

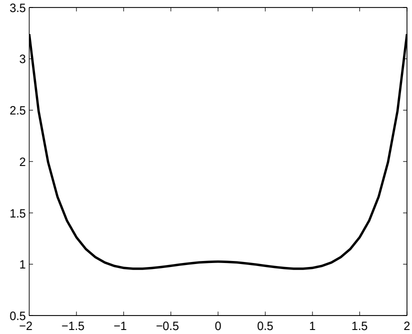

FIGURE 5.4: Ratio $f_{N(0,0.2635)}/f_U$.

in decomposition (4.2.3), we get

$$\begin{aligned} 0 \;\geq\; & n^{2/3}\frac{g(0)^2}{h_0(x_0)}\int_{[\,x_0,x_0+n^{-1/3}x)}\Big(\hat{F}_n(z)-F_0(z)\Big)\,dz \\ & -n^{2/3}\int_{[\,x_0,x_0+n^{-1/3}x)}\frac{g(z-x_0)}{h_0(z)}\,d(H_n-H_0)(z)+\sum_{i=1}^{5}n^{2/3}\tilde{R}_{n,i}(x)+A_n \end{aligned}$$

where $\tilde{R}_{n,i}(x) = R_{n,i}(x_0, x_0 + n^{-1/3}x), i = 1, \ldots, 5$, as in Section 4.2 and where equality holds for $x_0 + n^{-1/3}x \in \mathcal{T}_n$. Defining, for $x \geq 0$,

$$\tilde{R}_{n,6}(x) \;=\; \frac{g(0)^2}{h_0(x_0)}\int_{[\,x_0,x_0+n^{-1/3}x)}\big(F_0(x_0)-F_0(z)\big)\,dz+n^{-2/3}\frac{f_0(x_0)g(0)^2}{2h_0(x_0)}x^2$$

leads to

$$\begin{aligned} 0 \;\geq\; & n^{2/3}\frac{g(0)^2}{h_0(x_0)}\int_{[\,x_0,x_0+n^{-1/3}x)}\Big(\hat{F}_n(z)-F_0(x_0)\Big)\,dz \\ & -n^{2/3}\int_{[\,x_0,x_0+n^{-1/3}x)}\frac{g(z-x_0)}{h_0(z)}\,d(H_n-H_0)(z)-\frac{f_0(x_0)g(0)^2}{2h_0(x_0)}x^2 \\ & +\sum_{i=1}^{6}n^{2/3}\tilde{R}_{n,i}(x)+A_n, \end{aligned} \tag{5.1.4}$$

again with equality for $x_0 + n^{-1/3}x$ being a point of jump of $\hat{F}_n$.

Inequality (5.1.4) is the main ingredient of the proof of Theorem 5.1.3 and the following Lemmas state the behavior of the individual terms of the previous display. The behavior of $\tilde{R}_{n,i}$ as derived in Chapter 4, namely $n^{2/3}\tilde{R}_{n,i}(x) = O_p(1), i = 1, \ldots, 5$, is not sufficient in the present situation, $o_p(1)$ is needed rather than $O_p(1)$. Conjecture 5.1.2 is required to establish the needed faster rates for $\tilde{R}_{n,2}$ and $\tilde{R}_{n,4}$.

LEMMA 5.1.4.
Let x_0 be as in Assumption 5.1.1. Let

$$\mathcal{A}_n = \left\{ \alpha_{n,x} \,|\, \alpha_{n,x}(z) = n^{1/6}\frac{g(z-x_0)}{h_0(z)} \left(\mathbf{1}_{[\,x_0,\infty)}(z) - \mathbf{1}_{[\,x_0+n^{-1/3}x,\infty)}(z)\right),\ x \in \mathbb{R} \right\}$$

and define the processes $W_n(x) = \sqrt{n}\int \alpha_{n,x}(z)\, d(H_n - H_0)(z)$. Then in $l^\infty(\mathbb{R})$, the space of bounded functions $\nu : \mathbb{R} \to \mathbb{R}$, endowed with the metric of uniform convergence on compacta, W_n converges to a process W. This limit process is a mean-zero Gaussian process with covariance function $\operatorname{Cov}(W(t), W(s)) = \min(t,s)g(0)^2/h_0(x_0)$: a (rescaled) two-sided Brownian motion.

PROOF.
See Section 5.3. $\square$

LEMMA 5.1.5.
Choose x_0 according to Assumption 5.1.1. Then for all $c > 0$,

$$\sup_{x\in[\,-c,c\,]} n^{2/3}\left|\tilde{R}_{n,1}(x) + \tilde{R}_{n,3}(x) + \tilde{R}_{n,5}(x) + \tilde{R}_{n,6}(x)\right| \quad = \quad o_p(1).$$

Moreover, under Conjecture 5.1.2 we also have

$$\sup_{x\in[\,-c,c\,]} n^{2/3}\left|\tilde{R}_{n,2}(x) + \tilde{R}_{n,4}(x)\right| \quad = \quad o_p(1).$$

PROOF.
Let $c > 0$ be arbitrary and $\lambda > 0$ as in the Remark after Assumption 5.1.1 such that Lemma 4.2.2 (behavior of the remainder terms) and Theorem 4.3.1 (local rate of $\hat{F}_n$) hold.
Since $\sum_{i=1}^6 n^{2/3}\left|\tilde{R}_{n,i}(0)\right| = \sum_{i=1}^6 n^{2/3}\left|R_{n,i}(x_0,x_0)\right| = 0$, we restrict ourselves without loss of generality to $x \in (0, c\,]$. Then

$$\sup_{x\in(0,c\,]} n^{2/3}\left|\tilde{R}_{n,1}(x)\right| \quad = \quad \sup_{x\in(0,c\,]} n^{2/3}\frac{n^{-1/3}x}{n^{-1/3}x}\left|R_{n,1}(x_0, x_0+n^{-1/3}x)\right|. \qquad (5.1.5)$$

As a consequence of Lemma 4.4.4, by letting $s = x_0$ and $t = x_0 + n^{-1/3}x$, expression (5.1.5) can be bounded with probability tending to one by $n^{1/3}cMn^{-1/2}$ for some large M. The rate of $\tilde{R}_{n,2}$ is a direct consequence of Conjecture 5.1.2. Lemma 5.3.1 shows that $n^{2/3}\tilde{R}_{n,3}(x) = o_p(1)$ uniformly for $x \in (0, c\,]$. Using the same argument as in (5.1.5) together with Lemma 5.3.5, one gets

$$\sup_{x\in(0,c]} n^{2/3}\left|\tilde{R}_{n,4}(x)\right| \;=\; o_p(1).$$

Let $x_0-\lambda\le s<t\le x_0+\lambda$. Using the local rate result of Theorem 4.3.1 and replacing (4.4.9) in Lemma 4.4.9 (which derives the behavior of $R_{n,5}(s,t)$) by

$$\begin{aligned}
&\left|\int_{[s,t)}\left(\frac{g(0)}{\hat{h}_n(z)}-\frac{g(0)}{h_0(z)}\right)g(0)\left(\hat{F}_n(z)-F_0(z)\right)dz\right|\\
&\le \frac{g(0)^2}{c_\lambda}\int_{[s,t)}\left|\hat{h}_n(z)-h_0(z)\right|\left|\hat{F}_n(z)-F_0(z)\right|dz\\
&\le \frac{g(0)^2}{c_\lambda}\sup_{[0,\infty)}\left|\hat{h}_n(z)-h_0(z)\right|\sup_{[x_0-\lambda,x_0+\lambda]}\left|\hat{F}_n(z)-F_0(z)\right|(t-s),
\end{aligned}$$

which holds with probability tending to one, we obtain by using again $s=x_0$ and $t=x_0+n^{-1/3}x$ together with $t-s\le n^{-1/3}c$

$$\begin{aligned}
n^{2/3}\left|\tilde{R}_{n,5}(x)\right| &= n^{2/3}\left[(n^{-1/3}x)^{3/2}O_p(n^{1/3})+n^{-1/3}x\,o_p(n^{-1/3})\right]\\
&= n^{2/3}\left[O_p(n^{-5/6})+o_p(n^{-2/3})\right] \;=\; o_p(1)
\end{aligned}$$

uniformly for $x\in(0,c]$. Lemma 5.3.6 states the claimed behavior of $\tilde{R}_{n,6}$. □

LEMMA 5.1.6 (OFFSET TERM A_n).
The sequence (A_n) is tight, i.e. $n^{2/3}\left(\hat{C}_n(x_0)-\hat{C}_n(\tau^+)\right) \;=\; O_p(1).$

PROOF.
See Section 5.3. □

PROOF OF THEOREM 5.1.3.
Define for $x\in\mathbb{R}$

$$Y_n(x) \;:=\; n^{2/3}\frac{g(0)^2}{h_0(x_0)}\int_{[x_0,x_0+n^{-1/3}x)}\left(\hat{F}_n(x)-F_0(x_0)\right)dx-A_n$$

and

$$B_n(x) \;:=\; n^{2/3}\int_{[x_0,x_0+n^{-1/3}x)}\frac{g(z-x_0)}{h_0(z)}\,d(H_n-H_0)(z)+\frac{f(x_0)g(0)^2}{2h_0(x_0)}x^2+R_n(x)$$

with $R_n(x)=\sum_{i=1}^{6}\tilde{R}_{n,i}(x)$. Note that

(i) $Y_n(x) \le B_n(x)$, where equality holds if $x_0 + n^{-1/3}x$ is a point of jump of $\hat{F}_n$ (formula (5.1.4)),

(ii) A_n is tight and independent of x (Lemma 5.1.6),

(iii) $\sup_{[-c,c]} |R_n(x)| = \sup_{[-c,c]} |\sum_{i=1}^{6} \tilde{R}_{n,i}(x)| = o_p(1)$ for all $c > 0$ (Lemma 5.1.5).

By Lemma 5.1.4 and the asymptotic property of $R_n(x)$ we can now conclude that $\{B_n(x), x \in \mathbb{R}\}$ converges in $l^\infty(\mathbb{R})$ endowed with the metric of uniform convergence on compacta to the process $\{B(x), x \in \mathbb{R}\}$ given by

$$B(x) = \frac{g(0)}{\sqrt{h_0(x_0)}} W(x) + \frac{f(x_0)g(0)^2}{2h_0(x_0)} x^2.$$

Choosing

$$\begin{aligned} k_1 &= 2^{-1/3} f(x_0)^{1/3} g(0)^{-2/3} h_0(x_0)^{1/3} \text{ and} \\ k_2 &= 2^{2/3} f(x_0)^{-2/3} g(0)^{-2/3} h_0(x_0)^{1/3}, \end{aligned}$$

a Brownian scaling argument yields that the scaled processes $B_n^s(x) = k_1 B_n(k_2 x), x \in \mathbb{R}$, have a distributional limit process which is independent of any parameters related to g or F_0. To be precise $\{B_n^s(x),\, x \in \mathbb{R}\}$ converges in $l^\infty(\mathbb{R})$ to $\{B^s(x),\, x \in \mathbb{R}\}$ where

$$B^s(x) = k_1 \sqrt{k_2} \frac{g(0)}{\sqrt{h_0(x_0)}} W(x) + k_1 k_2^2 \frac{f(x_0)g(0)^2}{2h_0(x_0)} x^2 = W(x) + x^2. \tag{5.1.6}$$

We consider the same scaling to define the processes $Y_n^s(x) = k_1 Y_n(k_2 x)$. The inequality stated in (i) remains valid also for the scaled processes, i.e. $Y_n^s(x) \le B_n^s(x)$ for all $x \in \mathbb{R}$. Moreover, Y_n^s is a convex function below B_n^s that touches B_n^s if x is a point of jump of $\hat{F}_n$ and hence can be seen as the convex minorant of B_n^s. In addition

$$\begin{aligned} (Y_n^s)'(0) &= n^{1/3} k_1 k_2 \frac{g(0)^2}{h_0(x_0)} \left(\hat{F}_n(x_0) - F_0(x_0) \right) \\ &= n^{1/3} \left(\frac{2g(0)^2}{f(x_0)h_0(x_0)} \right)^{1/3} \left(\hat{F}_n(x_0) - F_0(x_0) \right). \end{aligned}$$

However, this, together with the convergence of B_n^s, does not as such imply that $(Y_n^s)'(0)$ converges in distribution to the derivative of the convex minorant of $W(x)+x^2$ evaluated at $x = 0$. The continuous mapping theorem using the derivative of the convex minorant functional can only be applied on compact intervals because the metric on $l^\infty(\mathbb{R})$ does not control the processes uniformly on $\mathbb{R}$. We therefore have to argue that the behavior of the derivative of the convex minorant of $B_n^s(x)$ and $W(x)+x^2$ at zero is with arbitrarily high probability determined by the processes on a sufficiently large interval $[-c, c]$ with $c > 0$.

Let us denote with $Y^s_{n,c}$ the convex minorant of B^s_n defined on $[-c, c]$. Then one can choose points of jump $\tau^- \in \mathcal{T}_n$ and $\tau^+ \in \mathcal{T}_n$ satisfying $\tau^+ - \tau^- = O_p(n^{-1/3})$ (see Lemma 4.3.5) such that $Y^s_n(\tau^-) = B^s_n(\tau^-)$ and $Y^s_n(\tau^+) = B^s_n(\tau^+)$. Hence we are able to choose for every $\mu > 0$ a $c > 0$ such that

$$\mathbb{P}\left(Y^s_n(x) = Y^s_{n,c}(x) : x \in [-1,1]\right) \quad \geq \quad 1 - \mu/2.$$

For the limit process we define Y^s_c as the convex minorant of $W(t) + t^2$ restricted to $[-c, c]$. Then we can argue the same way using Example 6.5 in Kim and Pollard (1990) that also Y^s_c coincides in a neighborhood of zero with the convex minorant of $W(x)+x^2$ on $\mathbb{R}$ with arbitrarily high probability. □

5.2 Conjecture 5.1.2

The estimator $\hat{F}_n$ attains a rate $n^{-1/3}$, globally and locally, as established in Theorems 4.1.1 and 4.1.4, respectively. Conjecture 5.1.2 characterizes specific functionals of $\hat{F}_n$ as smooth as it calls for a faster rate for converging to their true values than $\hat{F}_n$ itself, namely $n^{-1/2}$. In view of Theorem 5.1.3 we consider the deconvolution model specified in Assumption 5.1.1 and restrict the discussion in this section to the linear functional

$$K_t(F) \quad = \quad \int_{[0,t)} \big(g(t-x) - g(0)\big)\, dF(x),\ F \in \mathcal{F}_{[0,\infty)},$$

for $t \in (0, S_F)$. Deriving smoothness for K_t turned out to be a hard problem whose difficulties we will point out in this section together with an outline of an idea that is promising to lead to its proof.

The main idea for establishing the rate of K_t is to write

$$K_t(\hat{F}_n) - K_t(F_0) \quad = \quad \int_{[0,\infty)} b_{t,F_0}(z)\, d(H_n - H_0)(z) + R_{n,t} \tag{5.2.1}$$

for some function $b_{t,F_0} : [0, S_0 + S_g] \to \mathbb{R}$ (introduced and studied below) and a remainder term $R_{n,t}$, so that in the end, the rate $n^{-1/2}$ is deduced from the first term once we are able to control the remainder term $R_{n,t}$.

At present it seems that smoothness of K_t has to be derived separately for different choices of noise densities g and distribution functions F_0. We now present in the sequel a summary of the analysis for the so-called elbow deconvolution model, i.e. $g(y) = 2(1-y)\mathbb{1}_{[0,1]}(y)$, for $F_0 = U[0,1]$. This example illustrates the difficulties we encounter and a complete theory for this example is available in Groeneboom (2009). The analysis of other models is expected to run parallel to this one, though.

Let $t \in (0,1), F \in \mathcal{F}_{[\,0,\infty)}$, and denote with $L_2^0(P)$ the subset of functions contained in L_2 that have mean zero with respect a distribution function P. Note that $S_0 = S_g = 1$. The function $b_{t,F}$ is defined as the solution of the integral equation

$$\int_{[\,x,x+1\,]} b_{t,F}(z)g(z-x)\,dz \quad = \quad \big(g(t-x)-g(0)\big)\mathbf{1}_{[\,0,t)}(x) - K_t(F),\; x \in [\,0,1\,],$$

where $b_{t,F}$ has to have a representation

$$b_{t,F}(z) \quad = \quad \frac{\int_{[\,0,z\,]} g(z-x)\,dA_{t,F}(x)}{h_F(z)},\; z \in [\,0,2\,], \tag{5.2.2}$$

for some right continuous function $A_{t,F} : [\,0,1\,] \to \mathbb{R}$ with the property that $A_{t,F}(0) = 0$ and $A_{t,F}(x) = 0$ for all $x \geq 1$. Hence, combining the previous two displays, we have to study existence, uniqueness and properties of a solution $A_{t,F}$ of

$$\begin{aligned}\int_{[\,0,1\,]}\int_{x\vee v}^{1+x\wedge v} \frac{g(z-x)g(z-u)}{h_F(z)}\,dz\,dA_{t,F}(u)& \\ = \big(g(t-x)-g(0)\big)\mathbf{1}_{[\,0,t)}(x) - K_t(F).&\end{aligned} \tag{5.2.3}$$

Subsequently, (5.2.2) is used to define the corresponding function $b_{t,F}$.
Solving (5.2.3) for the underlying model, i.e. $F = F_0$, allows us to control the first term of the right side of (5.2.1). But we also have to solve (5.2.3) for $F = \hat{F}_n$ to derive the behavior of the remainder term $R_{n,t}$.

One difficulty for solving (5.2.3) is the requirement of $A_{t,F}$ having support $[\,0,1\,]$ which guarantees uniqueness of the corresponding $b_{t,F}$ and can therefore not be dropped. Also, even if we know that a (unique) solution exists, we cannot expect a closed expression for the functions $b_{t,F}$ for $F = F_0$ and $F = \hat{F}_n$ from which we can deduce the behavior of the function $b_{t,F}$. Hence, properties of $b_{t,F}$ have to be deduced directly from the integral equation using general integral equation theory.

We first transform (5.2.3) to an integral equation that will be the basis for the arguments presented here. Since the dependence on t is specific for the functional K_t, we omit it for the moment in the notation. It is shown in Groeneboom (2009) that equation (5.2.3) for the measure $dA_F = dA_{t,F}$ can be rewritten as an integral equation for the right-continuous function

$$A_F(x) \quad = \quad \int_{u\in(x,S_F\,]} dA_F(u) \;=\; -\int_{u\in[\,0,x\,]} dA_F(u),\; x \in [\,0,S_F\,].$$

However, we define $\tilde{A}_F(x) = A_F(x)/h_F(x)$ and state the integral equation in terms of $\tilde{A}_F$ (below). Note that A_F and $\tilde{A}_F$ behave similarly in a neighborhood of $x = 1$, but that $A_F(0) = 0$, while generally $\tilde{A}_F(0) \neq 0$. Working with $\tilde{A}$ has several advantages as we would know that A_F will have a bounded right derivative at 0 if we obtain a bounded solution for $\tilde{A}_F$. Also, the behavior of the derivative of $\tilde{A}_F$, which is crucial in the whole analysis, is simpler than the behavior of the derivative of A_F.

We first consider the underlying model, i.e. $F = F_0$. Let $k = k_t$ in general the right side of (5.2.3) depending on the functional. In case of K_t we have

$$k(x) \;=\; \big(g(t-x) - g(0)\big)\mathbf{1}_{[\,0,t)}(x) - K_t(F),\; x \in [\,0,1\,].$$

Then the integral equation for A_{F_0} can be written as

$$\tilde{A}_{F_0}(x) + \int_{u=0}^{1} \tilde{K}(x,u)\tilde{A}_{F_0}(u)\,du \;=\; \kappa(x),\;\; x \in [\,0,1\,], \tag{5.2.4}$$

where

$$\kappa(x) \;=\; \frac{k'(x)}{g(0)^2},\; x \in [\,0,1\,], \tag{5.2.5}$$

and

$$\begin{aligned}\tilde{K}(x,u) \;=\;& \frac{g'(x-u)h_F(u)}{g(0)h_F(x)}\,\mathbf{1}_{[0,x)}(u) + \frac{g'(u-x)}{g(0)}\,\mathbf{1}_{(x,1]}(u)\\ &+\frac{h_F(u)}{g(0)^2}\int_{z=x\vee u}^{1+x\wedge u} \frac{g'(z-x)g'(z-u)}{h_0(z)}\,dz. \end{aligned}\tag{5.2.6}$$

In fact, the kernel can be expressed as $\tilde{K}(x,u) = \tilde{K}_1(x,u) - \tilde{K}_2(x,u)$ with

$$\begin{aligned}\tilde{K}_1(x,u) \;&=\; \frac{h_F(u)}{g(0)^2}\int_{z=x\vee u}^{1+x\wedge u} \frac{g'(z-x)g'(z-u)}{h_0(z)}\,dz,\\ \tilde{K}_2(x,u) \;&=\; \frac{|g'(x-u)|h_F(u)}{g(0)h_F(x)}\,\mathbf{1}_{[0,x)}(u) + \frac{|g'(u-x)|}{g(0)}\,\mathbf{1}_{(x,1]}(u).\end{aligned}$$

For the elbow deconvolution model, we have

$$h_0(x) \;=\begin{cases} x(2-x) & ,\, x\in[0,1],\\ (2-x)^2 & ,\, x \in (1,2],\end{cases} \tag{5.2.7}$$

so that the formulas for the kernel boil down to

$$\tilde{K}_1(x,u) \;=\; h_0(u)\int_{z=x\vee u}^{1+x\wedge u} \frac{1}{h_0(z)}\,dz \;=\; h_0(u)\left\{\tfrac{1}{2}\log\left(\frac{2-x\vee u}{x\vee u}\right) + \frac{x\wedge u}{1-x\wedge u}\right\},$$

$$\tilde{K}_2(x,u) \;=\; -\frac{h_0(u)}{h_0(x\vee u)}.$$

The kernel $\tilde{K}_1$ is singular and the integral equation (5.2.4) can therefore not be handled by standard methods. We decompose the operator into two parts of which one is related to the singular part of $\tilde{K}_1$ and therefore 'catches' this singularity.

Definition 5.2.1 (Operators T_1 and T_2).
Let, using a slight abuse of notation, the operator $\tilde{K}$ be defined by

$$\big[\tilde{K}\phi\big](x) = \int_{u=0}^{1} \tilde{K}(x,u)\phi(u)\,du, \tag{5.2.8}$$

for functions $\phi : [0,1] \to \mathbb{R}$ such that the integral on the right-hand side of (5.2.8) is well-defined. The operator $\tilde{K}$ has the decomposition $\tilde{K} = T_1 + T_2$, where

$$\big[T_1\phi\big](x) \;=\; \int_{u=0}^{1} \phi(u)h_0(u)\int_{z=1}^{1+x\wedge u} \frac{dz}{h_0(z)}\,du \text{ and } T_2 \;=\; \tilde{K} - T_1. \tag{5.2.9}$$

The operator T_1 is related to the singular part of the kernel. We can now restrict the domain of functions on which we allow the integral operators to work on according to the following definition, where the singular part is singled out and handled separately. Moreover, it is shown in Groeneboom (2009) that, for the elbow deconvolution, the solution $\tilde{A}_{F_0}$ of the integral equation (5.2.4) belongs to the space $\mathcal{X}$.

Definition 5.2.2 (Spaces $\mathcal{X}, \mathcal{X}_0$ and $\mathcal{Y}$).
Let $\mathcal{X}_0$ be the normed space of right-continuous functions $\phi_0 : [\,0,1\,] \to \mathbb{R}$, having a fixed set $D \subset (0,1)$ of locations of possible discontinuities (corresponding to possible discontinuities of the derivative of κ). Moreover, assume that the functions $\phi_0 \in \mathcal{X}_0$ are left-continuous at $x = 1$, where we have $\phi_0(1) = 0$, and are continuously differentiable in $(0,1)\setminus D$, with a derivative ϕ_0' which can be extended to a bounded right-continuous function $\overline{\phi_0'}$ on $[0,1]$, by taking limits from the right at points of D, and taking the right limit at 0 and the left limit at 1. We endow $\mathcal{X}_0$ with the norm

$$\|\phi_0\|_{\mathcal{X}_0} \;=\; \sup_{x\in[\,0,1\,]} |\phi_0(x)| + \sup_{x\in[\,0,1\,]} \left|\overline{\phi_0'}(x)\right| + \sum_{x\in D} |\Delta\phi_0(x)|,$$

where $\Delta\phi_0(x)$ denotes $\phi_0(x) - \phi_0(x-)$.
Let the function $\phi_1 \in C[\,0,1\,]$ solve the equation

$$\big[(I+T_1)\phi_1\big](x) \;=\; 1,\, x \in [\,0,1\,]. \tag{5.2.10}$$

Then the space $\mathcal{X}$ *is defined by* $\mathcal{X} = \mathcal{X}_0 + \mathbb{R}\phi_1$ *with the norm* $\|\phi\|_{\mathcal{X}} = \|\phi_0\|_{\mathcal{X}_0} + |c|$ *for* $\phi = \phi_0 + c\phi_1$, *with* $\phi_0 \in \mathcal{X}_0$, $c \in \mathbb{R}$.
Moreover, let $\mathcal{Y}$ *be given by* $\mathcal{Y} = \{\psi : \psi = \psi_0 + c,\ \psi_0 \in \mathcal{X}_0,\ c \in \mathbb{R}\}$ *which is endowed with the norm* $\|\psi\|_{\mathcal{Y}} = \|\psi_0\|_{\mathcal{X}_0} + |c|$.

The function ϕ_1 is actually the solution in $C[\,0,1\,]$ of the integro-differential equation

$$\phi'(x) + \frac{1}{(1-x)^2}\int_{u=x}^{1} \phi(u)h_0(u)\,du \quad = \quad 0, \quad \phi(0) = 1. \tag{5.2.11}$$

It can be shown that ϕ_1 is asymptotically equivalent to $c_1(1-x)^{(-1+\sqrt{5})/2}$, as $x \uparrow 1$, with $c_1 \approx 1.1279$; illustrated in Figure 5.5. The function ϕ_1 is in fact a spherical Bessel function and a solution of the differential equation

$$(1-x)^2 y''(x) - 2(1-x)y'(x) - x(2-x)y(x) \quad = \quad 0,\ y(0) = 1,\ y(1) = 0.$$

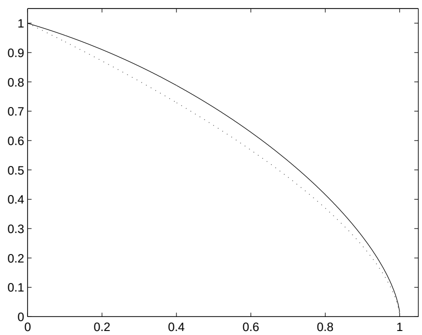

FIGURE 5.5: ϕ_1 (solid), $(1-x)^{(-1+\sqrt{5})/2}$ (dotted).

Note that due to Definition 5.2.1, equation (5.2.4), can be written as

$$(I + T_1)\tilde{A}_F(x) + T_2\tilde{A}_F(x) \quad = \quad \kappa(x),\ x \in [\,0,1\,]. \tag{5.2.12}$$

The following key lemma now described the properties of the operators $I + T_1$ and T_2 which actually make everything 'work'.

LEMMA 5.2.3 (PROPERTIES OF THE OPERATORS).
Let the spaces $\mathcal{X}, \mathcal{X}_0$ and $\mathcal{Y}$ be defined as in Definition 5.2.2. The operator $I+T_1$, defined on $\mathcal{X}$, has a bounded inverse $(I+T_1)^{-1}$ on the space $\mathcal{Y}$.
Also, the operator $T_2 : \mathcal{X} \to \mathcal{Y}$ is compact w.r.t. the norms $\|\cdot\|_{\mathcal{X}}$ on $\mathcal{X}$ and $\|\cdot\|_Y$ on $\mathcal{Y}$, respectively. Moreover, $T_2\phi \in C[0,1]$, if $\phi \in \mathcal{X}$.

With the help of this lemma one can reduce the integral equation (5.2.12) to an integral equation where the domain space and the range space, namely $\mathcal{X}$, are the same. We write

$$\tilde{A}_F + (I+T_1)^{-1}T_2\big(\tilde{A}_F\big) \quad = \quad (I+T_1)^{-1}(\kappa).$$

Then, due to Lemma 5.2.3 $(I+T_1)^{-1}(\kappa) \in \mathcal{X}$. By the same lemma, the operator T_2 is a compact mapping from $\mathcal{X}$ into $C[0,1] \cap \mathcal{Y}$, where $\mathcal{Y}$ is defined as in Definition 5.2.2. Hence, using Lemma 5.2.3 again, we get that $(I+T_1)^{-1}T_2$ is a compact mapping from $\mathcal{X}$ into $\mathcal{X}$, as the composition of a compact operator and a bounded operator.

To show that the integral equation has a unique solution we only have to prove that the homogeneous equation

$$\phi + \left[(I+T_1)^{-1}T_2\right](\phi) \quad = \quad 0 \tag{5.2.13}$$

has the trivial solution $\phi \equiv 0$ as its only solution (see, e.g., Kress (1989), Corollary 3.5, p. 29). This is proved in Groeneboom (2009) and gives the following result.

THEOREM 5.2.4 (CONVOLUTION OF ELBOW DENSITY WITH UNIFORM DENSITY).
Let $\kappa : [0,1] \to \mathbb{R}$ be given by $\kappa(x) = k'(x)/g(0)^2$ for $k \in L_2^0(F_0)$. Moreover, assume that $\kappa \in \mathcal{Y}$, where $\mathcal{Y}$ is defined as in Definition 5.2.2. Then integral equation (5.2.4)*, i.e*

$$\tilde{A}_{F_0}(x) + \int_{u=0}^{1} \tilde{K}(x,u)\tilde{A}_{F_0}(u)\,du \quad = \quad \kappa(x),\, x \in [0,1],$$

has a unique right-continuous solution $\tilde{A}_0$, such that $\tilde{A}_0(1) = 0$ and such that

$$|\tilde{A}_{F_0}(x)| = O\left((1-x)^{(-1+\sqrt{5})/2}\right),\, x \to 1.$$

For the purpose of illustrating the solution $\tilde{A}_{F_0}$, we first consider the 'mean functional' $K^1(F) = \int_{[0,\infty)} x\,dF(x)$ with $k(x) = x - K^1(F), x \in [0,1]$, implying $\kappa(x) = 1/4$, $x \in [0,1]$. A picture of $\tilde{A}_{F_0}$ as a solution of the equation (5.2.4) is shown in Figure 5.6. As mentioned above, $\tilde{A}_{F_0}$ is contained in $\mathcal{X}$ so that it is of some interest to see how this solution $\tilde{A}_{F_0}$ for the mean functional decomposes into a function $\phi_0 \in \mathcal{X}_0$ and the spherical Bessel function ϕ_1. In this case the solution $\tilde{A}_{F_0}$ is given by $\tilde{A}_{F_0} = \phi_0 + c\phi_1$,

$c \approx 0.464$ where ϕ_1 is illustrated in Figure 5.5 and ϕ_0 in Figure 5.7 (below). Note that the decomposition splits the solution into a negative part ϕ_0 and a positive part $c\phi_1$.

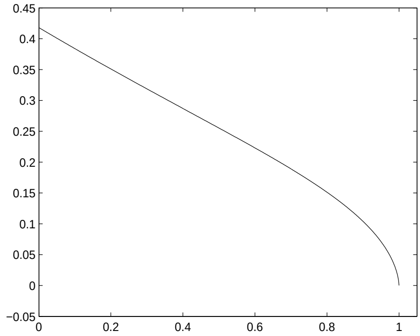

FIGURE 5.6: $\tilde{A}_0$ for $\kappa(x) \equiv 1/4$.

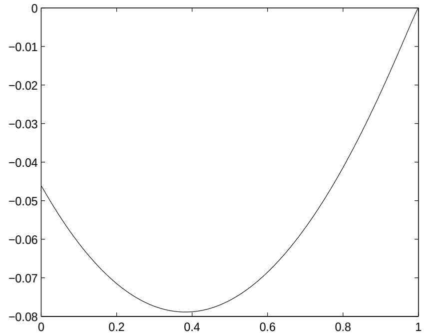

FIGURE 5.7: $\phi_0 = \tilde{A}_0 - 0.464\phi_1$, $\kappa \equiv 1/4$.

The picture of the solution $\tilde{A}_{F_0}$ for $\kappa(x) = \frac{1}{2}\mathbf{1}_{[\,0,t)}(x), x \in [\,0,1\,]$, which corresponds to $k(x) = \big(g(t-x) - g(0)\big)\mathbf{1}_{[\,0,t)}(x) - \int_{[\,0,t)} \big(g(t-x) - g(0)\big) dF_0(x)$ is shown in Figure 5.8.

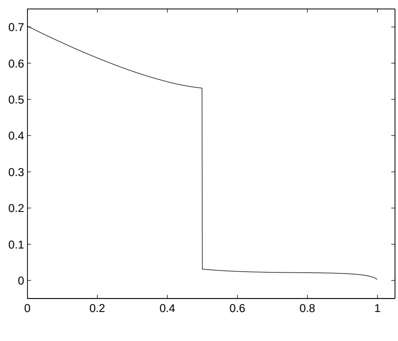

FIGURE 5.8: $\tilde{A}_0$ for $\kappa(x) = 0.5\mathbf{1}_{[\,0,0.5)}(x)$.

We now turn to the case $F = \hat{F}_n$ where an additional challenge is caused by the fact that $(Z_{(1)}, \hat{S}_n\,]$, with $Z_{(1)}$ being the first order statistic, almost surely does not coincide with the support of f_0: the first order statistic is almost surely larger than zero and $\hat{S}_n$ can be either smaller than 1 or larger than 1.

We start to solve for a right continuous function $A_{\hat{F}_n}$, defined on $[\,Z_{(1)},\hat{S}_n\,]$, which satisfies (recall that $\tilde{A}_F = A_F/h_F$)

$$\tilde{A}_{\hat{F}_n}(x) + \int_{u=Z_{(1)}}^{\hat{S}_n} \tilde{K}_n(x,u)\tilde{A}_{\hat{F}_n}(u)\,du \;=\; \kappa(x),\; x\in[\,Z_{(1)},\hat{S}_n), \tag{5.2.14}$$

under the constraint that

$$\int_{u\in[\,Z_{(1)},\hat{S}_n\,]} dA_{\hat{F}_n}(u) \;=\; 0, \tag{5.2.15}$$

where κ is defined by (5.2.5), and, for $x,u\in[\,Z_{(1)},\hat{S}_n)$,

$$\begin{aligned}\tilde{K}_n(x,u) \;=\; & \frac{g'(x-u)\hat{h}_n(u)}{g(0)\hat{h}_n(x)}\,\mathbf{1}_{[\,\hat{S}_n,x)}(u) + \frac{g'(u-x)}{g(0)}\,\mathbf{1}_{(x,\hat{S}_n\,]}(u)\\ & + \frac{\hat{h}_n(u)}{g(0)^2}\int_{z=x\vee u}^{1+x\wedge u}\frac{g'(z-x)g'(z-u)}{\hat{h}_n(z)}\,dz.\end{aligned} \tag{5.2.16}$$

Analogously to the treatment of the equations for the underlying model, we consider separately the part of the kernel which contains a singular part. On $D[\,Z_{(1)},\hat{S}_n)$, the space of 'cadlag'-functions on $[\,Z_{(1)},\hat{S}_n)$, we define the operators $T_{n,1}$ and $T_{n,2}$ by

$$\begin{aligned}[T_{n,1}\phi](x) \;&=\; \frac{1}{g(0)^2}\int_{u=Z_{(1)}}^{\hat{S}_n}\phi(u)\hat{h}_n(u)\int_{z=1}^{1+x\wedge u}\frac{g'(z-x)g'(z-u)}{\hat{h}_n(z)}\,dz\,du, && (5.2.17)\\ T_{n,2} \;&=\; \tilde{K}_n - T_{n,1},\end{aligned}$$

where $\tilde{K}_n$ is the integral operator corresponding to the kernel $\tilde{K}_n(x,u)$, analogously to the notation for the underlying model. In the case of the elbow deconvolution with the uniform density this becomes

$$[T_{n,1}\phi](x) \;=\; \int_{u=Z_{(1)}}^{\hat{S}_n}\phi(u)\hat{h}_n(u)\int_{z=1}^{1+x\wedge u}\frac{dz}{\hat{h}_n(z)}\,du,\; x\in[\,Z_{(1)},\hat{S}_n).$$

Strictly speaking, we would get

$$[T_{n,1}\phi](x) \;=\; \int_{u=Z_{(1)}}^{\hat{S}_n}\phi(u)\hat{h}_n(u)\int_{z=1\vee x}^{1+x\wedge u}\frac{dz}{\hat{h}_n(z)}\,du,\; x\in[\,Z_{(1)},\hat{S}_n)$$

from (5.2.17) (which is only relevant for the case $\hat{S}_n>1$), but to simplify matters, we take the lower bound 1 instead of $1\vee x$ in the inner integral for the elbow case.

Before defining the spaces on which the operators act, we consider the singular part first by considering the equivalent equation to (5.2.10).

LEMMA 5.2.5.
The equation

$$\phi(x) + [T_{n,1}\phi](x) \quad = \quad 1,\, x \in [\,Z_{(1)}, \hat{S}_n). \tag{5.2.18}$$

has a unique non-negative solution $\phi_{n,1}$ *in* $D[\,Z_{(1)}, \hat{S}_n)$. *We extend this function to a function in* $D[\,0,1\,]$ *by defining* $\phi_{n,1}(x) = \phi_{n,1}(Z_{(1)})$, $x \in [\,0, Z_{(1)})$ *and* $\phi_{n,1}(x) = 0$, $x \in [\,\hat{S}_n, 1\,]$, *in the case* $\hat{S}_n < 1$. *Then, in the elbow deconvolution model, the function* $\phi_{n,1}$ *converges almost surely uniformly in* $D[\,0,1\,]$ *to the function* ϕ_1, *defined by (5.2.10), which solves the equation* $(I + T_1)\phi = 1$, $x \in [\,0,1\,]$ *in the underlying model.*

Using a sample size $n = 1000$ with $\hat{S}_n = 0.88983 < 1$, Figure 5.9 shows the solution $\phi_{1000,1}$ of (5.2.18) together with the corresponding function ϕ_1 (see Figure 5.5) for the underlying model.

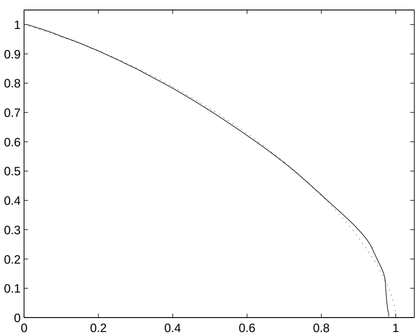

FIGURE 5.9: $\phi_{1000,1}$ (solid), ϕ_1 (dotted); $\hat{S}_{1000} = 0.88983$.

Next we have to define the relevant spaces on which the random equations are solved.

DEFINITION 5.2.6 (DEFINITION $\mathcal{X}_{n,0}, \mathcal{X}_n$ AND $\mathcal{Y}_n$).
Let $\mathcal{X}_{n,0}$ *be the normed space of bounded right-continuous functions* $\phi_0 : [\,Z_{(1)}, \hat{S}_n\,] \to \mathbb{R}$, *such that* ϕ_0 *has only two types of discontinuities:*

(i) discontinuities at points of jump of $\hat{F}_n$; *we call this set* $D_{n,1}$.

(ii) a finite set of discontinuities in $[\,Z_{(1)}, \hat{S}_n)$, *corresponding to points of jump of* κ; *we call this set* $D_{n,2}$. *We assume that this set of possible discontinuities is bounded, which is the same as that for the underlying model, see Definition 5.2.2.*

Define $D_n = D_{n,1} \cup D_{n,2}$. *Moreover, assume that the functions* $\phi_0 \in \mathcal{X}_{n,0}$ *are continuously differentiable in* $(Z_{(1)}, \hat{S}_n) \setminus D_n$, *and that the derivative* ϕ_0' *can be extended to a bounded right-continuous function* $\overline{\phi_0'}$, *defined on* $[\, Z_{(1)}, \hat{S}_n\,]$. *Then we define*

$$\|\phi_0\|_{\mathcal{X}_{n,0}}$$
$$= \sup_{x\in[\,Z_{(1)},\hat{S}_n\,]} |\phi_0(x)| + \sup_{x\in[\,Z_{(1)},\hat{S}_n\,]} |\overline{\phi_0'}(x)| + \sup_{x\in D_{n,1}} \frac{|\Delta\phi_0(x)|}{\Delta\hat{F}_n(x)} + \sum_{x\in D_{n,2}} |\Delta\phi_0(x)|,$$

with $\|\phi_0\|_{\mathcal{X}_{n,0}} < \infty$ *and where* $\Delta\phi_0(x)$ *and* $\Delta\hat{F}_n(x)$ *denote the jumps of* ϕ_0 *and* $\hat{F}_n$ *at* x, *respectively, and* $\|\phi_0\|_{\mathcal{X}_{n,0}}$ *denotes the norm of* ϕ_0 *in* $\mathcal{X}_{n,0}$.
Let the space $\mathcal{X}_n$ *be defined by* $\mathcal{X}_n = \mathcal{X}_{n,0} + \mathbb{R}\phi_{n,1}$, *where* $\phi_{n,1}$ *is the solution of the equation (5.2.18). The space* $\mathcal{X}_n$ *possesses the norm* $\|\phi\|_{\mathcal{X}_n} = \|\phi_0\|_{\mathcal{X}_{n,0}} + |c|$ *for* $\phi = \phi_0 + c\phi_{n,1}$ *with* $\phi_0 \in \mathcal{X}_{n,0}$, $c \in \mathbb{R}$.
Finally, $\mathcal{Y}_n$ *is defined by* $\mathcal{Y}_n = \{\psi : \psi = \psi_0 + c,\, \psi_0 \in \mathcal{X}_{n,0},\, c \in \mathbb{R}\}$, *which is endowed with the norm* $\|\psi\|_{\mathcal{Y}_n} = \|\psi_0\|_{\mathcal{X}_{n,0}} + |c|$, $\psi = \psi_0 + c$, $\psi_0 \in \mathcal{X}_{n,0}$, $c \in \mathbb{R}$.

We now have the following lemma, similar to Lemma 5.2.3.

Lemma 5.2.7.
Let the spaces $\mathcal{X}_n$ *and* $\mathcal{Y}_n$ *be defined as in Definition 5.2.6. Then the operator* $I + T_{n,1}$, *defined on* $\mathcal{X}_n$, *has a bounded inverse* $(I + T_{n,1})^{-1}$ *on the space* $\mathcal{Y}_n$.

Analogously to the situation for the underlying model, the mapping

$$(I + T_{n,1})^{-1} T_{n,2}$$

is a compact mapping from $\mathcal{X}_n$ into $\mathcal{X}_n$, and the integral equation

$$\phi + \big[(I + T_{n,1})^{-1} T_{n,2}\phi\big](x) \quad = \quad (I + T_{n,1})^{-1}(\kappa)$$

has a unique solution ϕ_n in $\mathcal{X}_n$.

The solution $\tilde{A}_{\hat{F}_n}$, together with the solution $\tilde{A}_{F_0}$ for the underlying model (see also Figure 5.8) is shown in Figure 5.10 ($n = 1000, \hat{S}_{1000} = 1.06661$).

Recall that in the elbow convolution model for the specific functional $K_t(F)$, the function k is given by

$$k(x) \quad = \quad \big(g(t-x) - g(0)\big)\mathbf{1}_{[\,0,t)}(x) - \int_{[\,0,t)} \big(g(t-x) - g(0)\big)\, dF_0(x),\; x \in [\,0,1\,],$$

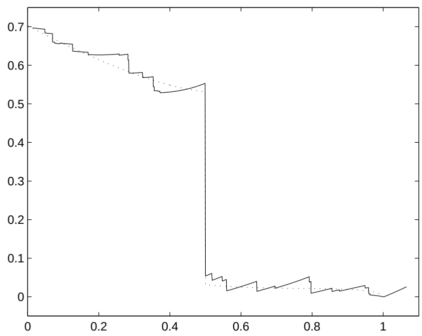

FIGURE 5.10: $\tilde{A}_{\hat{F}_{1000}}$ (solid), $\tilde{A}_{F_0}$ (dotted); $t = 0.5$.

which leads in the elbow deconvolution model to

$$\kappa(x) \quad = \quad \tfrac{1}{2}\mathbf{1}_{[\,0,t)}(x),\; x \in [\,0, \hat{S}_n\,],$$

with $\kappa'(x) = 0$, $x \in [\,0, \hat{S}_n\,] \setminus \{t\}$ and $\Delta\kappa(t) = -\frac{1}{2}$. The function $\phi = (I + T_1)^{-1}(\kappa)$ is of the form $\phi = \phi_0 - c\phi_1$ for

$$c \quad = \quad \phi_0(0) - 0.5 = \big[T_1\phi_0\big](1) = \int_{u=0}^{t} \frac{u\phi_0(u)h_0(u)}{1-u}\,du \approx 0.0559023,$$

where ϕ_1 is defined by (5.2.10). A picture of ϕ_0 (made in Mathematica) is shown in Figure 5.11 and a picture of ϕ, arising from numerically solving the integral equation (see Groeneboom, 2009), in Figure 5.12.

For the random equations, involving the MLE $\hat{F}_n$, we get similarly the integral equation $(I + T_{n,1})\phi = \kappa$, solved by a function ϕ_n (see Figure 5.14, $n = 1000, \hat{S}_{1000} = 0.88983$) of the form $\phi_n = \phi_{n,0} - c_n\phi_{n,1}$ for $\phi_{n,0} \in \mathcal{X}_{n,0}$ (see Figure 5.13), $c_n \in \mathbb{R}$, and $\phi_{n,1}$ as defined in (5.2.18).

It is shown in Groeneboom (2009) that $\phi_{n,0} \to \phi_0$ and $c_n \to c$ with probability one, where ϕ_0 and c are the corresponding constant and function in the underlying model, respectively. Since $\phi_{n,1}$ converges uniformly to ϕ_1 in the supremum metric on $D[\,0,1\,]$, we have the following theorem.

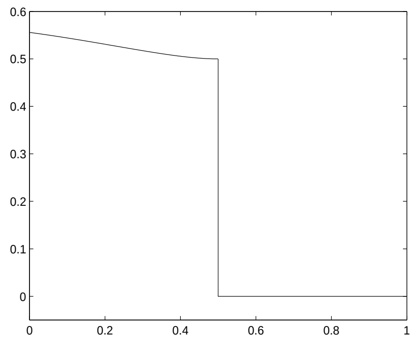

FIGURE 5.11: ϕ_0, $\kappa = 0.5\mathbf{1}_{[0,0.5)}$; $\phi_0(x) = 0$ for $x \geq 0.5$.

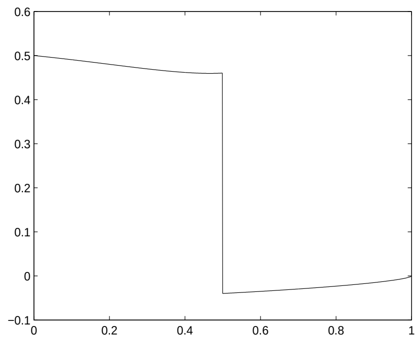

FIGURE 5.12: $\phi = (I + T_1)^{-1}(\kappa)$ for $\kappa = 0.5\mathbf{1}_{[0,0.5)}$.

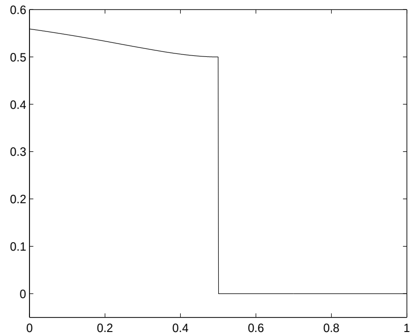

FIGURE 5.13: $\phi_{n,0}$, $\kappa = 0.5\mathbf{1}_{[0,0.5)}$; $\phi_{n,0}(x) = 0$ for $x \geq 0.5$.

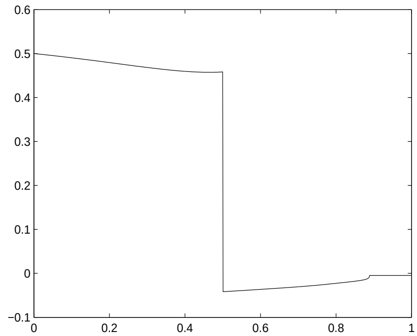

FIGURE 5.14: $\phi_n = (I + T_{n,1})^{-1}(\kappa)$ for $\kappa = 0.5\mathbf{1}_{[0,0.5)}$.

THEOREM 5.2.8.
Let, in the elbow deconvolution model, ϕ_n solve the equation

$$(I + T_{n,1})\phi_n \quad = \quad \mathbf{1}_{[\,0,t)},\ \phi_n = \phi_{n,0} - c_n\phi_{n,1},$$

where $\phi_{n,0} \in \mathcal{X}_{n,0}$, and $\phi_{n,1}$ is defined as in Lemma 5.2.5. Then ϕ_n converges almost surely, in the supremum metric on $D[\,0,1\,]$, to the function ϕ solving

$$(I + T_1)\phi \quad = \quad \mathbf{1}_{[\,0,t)}$$

for the underlying model.

Combining all these results we find that the functions $A_{\hat{F}_n}$ are of uniformly bounded variation, and converge almost surely to A_{F_0}, using the norms of the spaces $\mathcal{X}_n$ and $\mathcal{X}$.

Introducing the dependence on t (which will be important in the remainder of this section) by writing $A_{t,F_0} = A_{F_0}$ and $A_{t,\hat{F}_n} = A_{\hat{F}_n}$, we now define for $z \in [0,2]$ the functions b_{t,F_0} and $b_{t,\hat{F}_n}$. First we have, according to (5.2.2),

$$b_{t,F_0}(z) = \frac{\int_{x\in[0,z]} g(z-x)\,dA_{F_0}(x)}{h_0(z)},\ z \in [0,2].$$

It is shown in Groeneboom (2009) that

$$0 \le b_{t,F_0}(z) \le c(2-z)^{(-3+\sqrt{5})/2} \approx c(2-z)^{-0.38197},\ z \uparrow 2,$$

for a constant $c > 0$, implying that $b_{t,F_0}(z)\sqrt{h_0(z)} \to 0$ as $z \uparrow 2$ which, in turn, implies $b_{t,F_0} \in L_2(H_0)$. In Figure 5.15 we show the function $b_{0.5,F_0}$ on $[0,2]$, scaled by $B(z) = \left(\sqrt{h_0(z)} + \sqrt{\hat{h}_n(z)}\right)/2$.

For the random function we have

$$b_{t,\hat{F}_n}(z) = \frac{\int_{x\in[0,z]} g(z-x)\,dA_{t,\hat{F}_n}(x)}{\hat{h}_n(z)},\ z \in [Z_{(1)}, \hat{S}_n). \tag{5.2.19}$$

In order to proceed with the general argument, we require that $b_{t,\hat{F}_n}$ is -just as b_{t,F_0}- defined on $[0,2]$. Hence, we want to extend $b_{t,\hat{F}_n}$ on $[0, Z_{(1)})$ and if $\hat{S}_n < 1$ also on $[1+\hat{S}_n, 2]$. For that we cannot use relation (5.2.19) as $\hat{h}_n(z) = 0$ for all $z \in [0, Z_{(1)}) \cup [1+\hat{S}_n, 2]$ (the latter interval again just for $\hat{S}_n < 1$). In Groeneboom (2009) an extension $b_{t,\hat{F}_n}$ on $[0,2]$ is derived such that

$$\int_{z=x}^{1+x} b_{t,\hat{F}_n}(z) g(z-x)\,dz = k(x) - \int k(u)\,d\hat{F}_n(u),\ x \in [0,1]. \tag{5.2.20}$$

The resulting function is shown in Figure 5.15 ($n = 1000, \hat{S}_{1000} = 0.966962$), again scaled by $B(z) = \left(\sqrt{h_0(z)} + \sqrt{\hat{h}_n(z)}\right)/2$ which shows a remarkably good fit between $b_{t,F_0}(z)$ and $b_{t,\hat{F}_n}(z)$.

All this means that in fact the proof of the conjectured behavior of K_t can be possibly completed along the following line.

(I) Relate $K_t(\hat{F}_n) - K_t(F_0)$ to an empirical process with respect to H_n.

(a) Due to Lemma 5.3.7 we have

$$K_t(\hat{F}_n) - K_t(F_0) = -\int_{[0,2]} b_{t,\hat{F}_n}(z)\,dH_0(z). \tag{5.2.21}$$

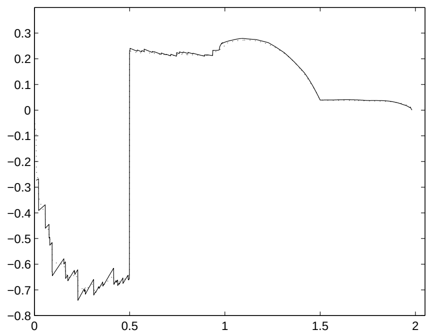

FIGURE 5.15: $B \cdot b_{0.5,\hat F_{1000}}$ (solid), $B \cdot b_{0.5,F_0}$ (dotted).

(b) We can define a piecewise constant, right-continuous version $\bar A_{t,\hat F_n}$ of $A_{t,\hat F_n}$ such that $\bar A_{t,\hat F_n}$ is equal to $A_{t,\hat F_n}$ at points of jump of $\hat F_n$. Let

$$\bar b_{t,\hat F_n}(z) \quad = \quad \frac{\int_{u\in[\,0,z\,]} g(z-u)\,d\bar A_{t,\hat F_n}(u)}{\hat h_n(z)}\,,\ z \in [\,Z_{(1)}, \hat S_n),$$

where we define $\bar b_{t,\hat F_n}(z)$ to be zero if $\hat h_n(z) = 0$. Note that, by Lemma 5.3.8,

$$\int \bar b_{t,\hat F_n}(z)\,dH_n(z) \quad = \quad 0. \tag{5.2.22}$$

(c) Incorporate (5.2.22) into (5.2.21) using the decomposition

$$\begin{aligned}
K_t(\hat F_n) - K_t(F_0) \;&=\; -\int b_{t,\hat F_n}(z)\,dH_0(z)\\
&= \; -\int \bar b_{t,\hat F_n}(z)\,dH_0(z) + \int \Big(\bar b_{t,\hat F_n}(z) - b_{t,\hat F_n}(z)\Big)\,dH_0(z)\\
&= \; \int b_{t,F_0}(z)\,d(H_n - H_0)(z) + \int \Big(\bar b_{t,\hat F_n}(z) - b_{t,F_0}(z)\Big)\,d(H_n - H_0)(z)\\
&\quad + \int \Big(\bar b_{t,\hat F_n}(z) - b_{t,\hat F_n}(z)\Big)\,dH_0(z) - \int \bar b_{t,\hat F_n}(z)\,dH_n(z), \qquad (5.2.23)
\end{aligned}$$

where $b_{t,\hat F_n}$ is extended to the interval $[\,0,2\,]$.

(II) Study the terms appearing in (5.2.23) separately.

(a) The first term of the right side of (5.2.23), multiplies by $\sqrt{n}$, will converge to a normal distribution with expectation zero and variance $\int b_{t,F_0}(z)^2\,dH_0(z)$. The preceding results showed that this (efficient) variance is indeed finite in the example considered.

(b) The above discussion also shows that the second term is $o_p(n^{-1/2})$ for the elbow deconvolution model. This computation is rather subtle, though, since $b_{t,F_0}(z) \to \infty$, as $z \uparrow 2$, and $b_{t,\hat{F}_n}(z)$ will also not be uniformly bounded near 2, by the almost sure pointwise convergence of the function $\phi_{n,1}$ to ϕ_1 (see above). Here the decomposition of the differentiable part of the solution into a function with a derivative of bounded variation and the derivative of the function $\phi_{n,1}$, defined in Lemma 5.2.5, plays an essential role.

(c) Argue that the third term of (5.2.23) is of order $O_p(n^{-1/2})$ by first writing (see Lemma 5.3.9)

$$\int \left(\bar{b}_{t,\hat{F}_n}(z) - b_{t,\hat{F}_n}(z)\right) dH_0(z)$$
$$= \int \left(\bar{b}_{t,\hat{F}_n}(z) - b_{t,\hat{F}_n}(z)\right)\left(h_0(z) - \hat{h}_n(z)\right) dz.$$

We can now apply the Cauchy-Schwarz inequality, yielding

$$\left|\int \left(\bar{b}_{t,\hat{F}_n}(z) - b_{t,\hat{F}_n}(z)\right)\left(h_0(z) - \hat{h}_n(z)\right) dz\right|$$
$$\le \left\|\left(\bar{b}_{t,\hat{F}_n} - b_{t,\hat{F}_n}\right)\left(\sqrt{h_0} + \sqrt{\hat{h}_n}\right)\right\|_{L_2} \left\|\frac{h_0 - \hat{h}_n}{\sqrt{h_0} + \sqrt{\hat{h}_n}}\right\|_{L_2},$$

where we hope to show

$$\left\|\left(\bar{b}_{t,\hat{F}_n} - b_{t,\hat{F}_n}\right)\left(\sqrt{h_0} + \sqrt{\hat{h}_n}\right)\right\|_{L_2} \left\|\frac{h_0 - \hat{h}_n}{\sqrt{h_0} + \sqrt{\hat{h}_n}}\right\|_{L_2}$$
$$\le c_1 \left\|\hat{F}_n - F_0\right\|_{L_2} \left\|\sqrt{\hat{h}_n} - \sqrt{h_0}\right\|_{L_2} = O_p\left(n^{-2/3}\right)$$

for a constant $c_1 > 0$. Note that we can write, for $c_2 > 0$,

$$\left\|\left(\bar{b}_{t,\hat{F}_n}(z) - b_{t,\hat{F}_n}(z)\right)\left(\sqrt{h_0} + \sqrt{\hat{h}_n}\right)\right\|_{L_2}$$
$$\le c_2 \left\|\bar{b}_{t,\hat{F}_n}(z) - b_{t,\hat{F}_n}(z)\right\|_{h_0+\hat{h}_n},$$

where

$$\|\phi\|_{h_0+\hat{h}_n}^2 = \int_{z=0}^{2} \phi(z)^2 \left(\hat{h}_n(z) + h_0(z)\right) dz,\ \phi \in L_{2,h_0+\hat{h}_n},$$

which is a natural norm to use, since $b_{t,\hat{F}_n}$ minimizes the norm $\|b\|_{\hat{h}_n}$ and b_{t,F_0} minimizes the norm $\|b\|_{h_0}$ in the allowed class of functions.
The line of argument is similar to that in Geskus and Groeneboom (1996) and Geskus and Groeneboom (1997), but also more complicated, because of the behavior of $b_{t,\hat{F}_n}(z)$ and $\bar{b}_{t,\hat{F}_n}(z)$ for z in a neighborhood of 2.

A picture of $b_{t,\hat{F}_n}$ and its piecewise constant version $\bar{b}_{t,\hat{F}_n}$, multiplied by

$$C(z) \;=\; \mathbf{1}_{[0,1)}(z) + \frac{\sqrt{h_0} + \sqrt{\hat{h}_n}}{\sqrt{h_0(1)} + \sqrt{\hat{h}_n(1)}} \mathbf{1}_{[1,2]}(z), \tag{5.2.24}$$

is shown in Figure 5.16 and Figure 5.17 ($n = 10000$, $\hat{S}_{10000} = 0.98211$). Note the closeness of the rescaled $b_{t,\hat{F}_n}$ and $\bar{b}_{t,\hat{F}_n}$ on $[0,2]$. Moreover, $\bar{b}_{t,\hat{F}_n}$ does not have a jump at $t = 0.5$, but at $\tau = 0.50258$ which is the location of the closest point of jump of $\hat{F}_n$ to the right of t.

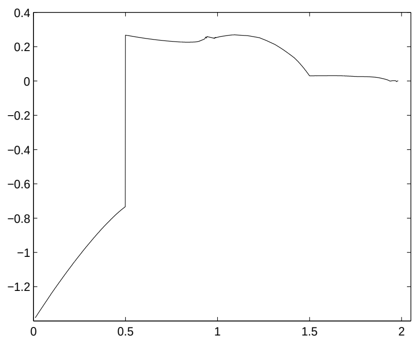

FIGURE 5.16: $C \cdot b_{0.5,\hat{F}_{10000}}$ (solid) with C as in (5.2.24).

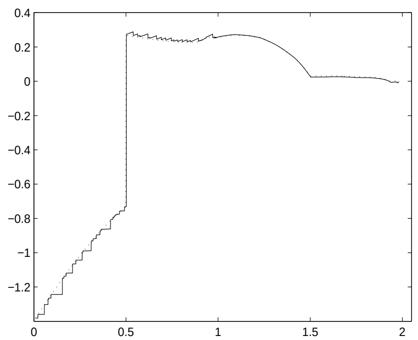

FIGURE 5.17: $C \cdot b_{0.5,\hat{F}_{10000}}$ (dotted), $C \cdot \bar{b}_{0.5,\hat{F}_{10000}}$ (solid).

5.3 PROOFS AND TECHNICAL LEMMAS

PROOF OF LEMMA 5.1.4.
Let $c > 0$. The Lemma follows from Theorem 2.11.22 and Example 2.11.24 in van der Vaart and Wellner (1996) which state conditions for a limit theorem for processes indexed by classes of functions that change with n. Let $c_\lambda > 0$ according to the remark after Assumption 5.1.1 such that $h_0(z) \geq c_\lambda$ for all $z \in [x_0 - n^{-1/3}c, x_0 + n^{-1/3}c]$ and n sufficiently large.

Define for each n the envelope function $A_n(x) = n^{1/6} g(0) c_\lambda^{-1} \mathbf{1}_{[x_0, x_0+n^{-1/3}c]}(x)$. Then the conditions of the mentioned Theorem 2.11.22 are met since $\mathcal{A}_n$ is a Vapnik Chervonenkis-class (VC-class) with VC-index $V(\mathcal{A}_n) = 2 < \infty$ independent of n and

$$\int_{[0,\infty)} A_n(z)^2 \, dH_0(z) \;\leq\; n^{1/3} \frac{g(0)^3}{c_\lambda^2} n^{-1/3} c \;=\; O(1).$$

In addition, A_n satisfies, for every $\eta > 0$,

$$\int_{[0,\infty)} A_n^2(z)\mathbf{1}_{\{A_n > \eta\sqrt{n}\}}(z)\, dH_0(z) \quad \rightarrow \quad 0,\ n \to \infty,$$

since $\mathbf{1}_{n^{1/6} g(0) c_\lambda^{-1} > \eta n^{1/2}} = 0$ for $n \geq \big(g(0)\eta^{-1}c_\lambda^{-1}\big)^3$. Assume $0 < s < t$. Then

$$\sup_{t-s<\delta_n} \int_{[0,\infty)} \big(\alpha_{n,s}(z) - \alpha_{n,t}(z)\big)^2\, dH_0(z) \quad \rightarrow \quad 0 \text{ for every } \delta_n \downarrow 0$$

due to

$$\begin{aligned} 0 \quad &\leq \quad \sup_{t-s<\delta_n} \int_{[0,\infty)} \big(\alpha_{n,s}(z) - \alpha_{n,t}(z)\big)^2\, dH_0(z) \\ &\leq \quad \sup_{t-s<\delta_n} \frac{g(0)^2}{c_\lambda} n^{1/3}(n^{-1/3}t - n^{-1/3}s) \leq \frac{g(0)^2}{c_\lambda}\delta_n \rightarrow 0 \text{ for } \delta_n \downarrow 0. \end{aligned}$$

Moreover, $\operatorname{Cov}(\alpha_{n,s}, \alpha_{n,t}) \to s g(0)^2/h_0(x_0)$, $n \to \infty$, since, with $G(x) = \int_0^x g(t)\, dt$,

$$\begin{aligned} &\int_{[0,\infty)} \alpha_{n,s}(z)\alpha_{n,t}(z)\, dH_0(z) - \int_{[0,\infty)} \alpha_{n,s}(z)\, dH_0(z) \int_{[0,\infty)} \alpha_{n,t}(z)\, dH_0(z) \\ &= \quad \int_{x_0}^{x_0+n^{-1/3}s} n^{1/3}\frac{g(z-x_0)^2}{h_0(z)}\, dz - s\,\frac{G(n^{-1/3}s) - G(0)}{n^{-1/3}s} \cdot \big(G(n^{-1/3}t) - G(0)\big) \\ &\rightarrow \quad s\frac{g(0)^2}{h_0(x_0)} - s g(0)\cdot 0. \quad \square \end{aligned}$$

LEMMA 5.3.1 ($\tilde{R}_{n,3}$).
Under Assumption 5.1.1, for $c > 0$,

$$\sup_{x\in[0,c]} n^{2/3} \int_{[x_0, x_0+n^{-1/3}x)} g(z-x_0)\left(\frac{1}{\hat{h}_n(z)} - \frac{1}{h_0(z)}\right) d(H_n - H_0)(z) \quad = \quad o_p(1).$$

PROOF.
Let $c > 0$. For small positive γ define the index set $\mathcal{I}^\gamma$

$$\mathcal{I}^\gamma \quad = \quad \{F \in \mathcal{F}_{[0,\infty)} : \sup_{[0,\infty)} |F(x) - F_0(x)| < \gamma\}$$

and, using $\mathcal{I}^\gamma$, the function class

$$\begin{aligned} \mathcal{H}_n^\gamma = \Big\{h_{n,(F,x)}^\gamma \,\Big|\, h_{n,(F,x)}^\gamma(z) \;=\; n^{1/6} g(z-x_0)\left(\frac{1}{h_F(z)} - \frac{1}{h_0(z)}\right)\mathbf{1}_{[x_0, x_0+n^{-1/3}x)}(z) : \\ (F,x) \in \mathcal{I}^\gamma \times [0,c],\ z \geq 0\Big\}. \qquad (5.3.1) \end{aligned}$$

Note that the elements of $\mathcal{H}_n^\gamma$ are well defined functions for $\gamma \le c_0/(2g(0))$, where c_0 denotes the lower bound of h_0 on $[\,x_0, x_0 + S_g\,]$ according to Lemma 4.2.1. To see this, note that $h_F(z) \ge h_0(z) - 2g(0)\gamma > 0$ for all $z \in [\,x_0, x_0 + S_g\,]$ by the choice of γ.

Note also that for every $\gamma > 0$ the set $\mathcal{I}^\gamma$ contains $\hat{F}_n$ with arbitrarily high probability if n is taken sufficiently large due to its uniform consistency (see Theorem 3.1.5). Hence,

$$\begin{aligned}
&\sup_{x\in[\,0,c\,]} \left| n^{2/3} \int_{[\,x_0, x_0+n^{-1/3}x)} g(z-x_0)\left(\frac{1}{\hat{h}_n(z)} - \frac{1}{h_0(z)}\right) d(H_n - H_0)(z) \right| \\
&\le \sup_{\mathcal{H}_n^\gamma} n^{1/2} \left| \int_{[\,0,\infty)} h_{n,(F,x)}^\gamma(z)\, d(H_n - H_0)(z) \right|
\end{aligned}$$

holds with arbitrarily high probability for sufficiently large n. The previous display implies that it suffices to show that for all $\sigma > 0$

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{\mathcal{H}_n^\gamma} n^{1/2} \left| \int_{[\,0,\infty)} h_{n,(F,x)}^\gamma(z)\, d(H_n - H_0)(z) \right| > \sigma \right) = 0. \tag{5.3.2}$$

Let $\sigma > 0$. Applying Markov's inequality to the probability in (5.3.2) and the maximal inequality for empirical processes of Lemma 19.34 of van der Vaart (1998) afterwards, yields

$$\begin{aligned}
&\mathbb{P}\left(\sup_{\mathcal{H}_n^\gamma} \left| \sqrt{n} \int_{[\,0,\infty)} h_{n,(F,x)}^\gamma(z)\, d(H_n - H_0)(z) \right| > \sigma \right) \\
&\le \frac{1}{\sigma}\, \mathbb{E}_{H_0}\left(\sup_{\mathcal{H}_n^\gamma} \left| \sqrt{n} \int_{[\,0,\infty)} h_{n,(F,x)}^\gamma(z)\, d(H_n - H_0)(z) \right| \right) \\
&\le \frac{1}{\sigma}\sqrt{n} \int_{[\,0,\infty)} H_n^\gamma(z) \mathbf{1}_{H_n^\gamma > \sqrt{n} a(\kappa_\gamma)}(z)\, dH_0(z) + \frac{1}{\sigma} J_{[\,]}(\kappa_\gamma, \mathcal{H}_n^\gamma, L_2(H_0))
\end{aligned} \tag{5.3.3}$$

where H_n^γ denotes an envelope function of $\mathcal{H}_n^\gamma$, where $\kappa_\gamma \ge \|h_{n,(F,x)}^\gamma\|_{L_2(H_0)}$ for all $h_{n,(F,x)}^\gamma \in \mathcal{H}_n^\gamma$, where $a(\kappa_\gamma) = \kappa_\gamma / \sqrt{1 + H_{[\,]}(\kappa_\gamma, \mathcal{H}_n^\gamma, L_2(H_0))}$ and where $J_{[\,]}(\delta, \mathcal{H}_n^\gamma, L_2(H_0)) = \int_0^\delta \sqrt{H_{[\,]}(\tau, \mathcal{H}_n^\gamma, L_2(H_0))}\, d\tau$.

Let $\gamma < c_0/(4g(0))$ and n sufficiently large such that $[\,x_0, x_0 + n^{-1/3}c\,] \subset [\,x_0, x_0 + S_g\,]$. Then by the choice of γ, every h_F with $F \in \mathcal{I}^\gamma$ is bounded below on $[\,x_0, x_0 + S_g\,]$ by $c_0/2$ since $h_F(z) \ge h_0(z) - 2g(0)\gamma > c_0 - c_0/2$ for all $z \in [\,x_0, x_0 + S_g\,]$.

The envelopes H_n^γ are therefore chosen as

$$H_n^\gamma(z) = n^{1/6}\, c_h\, \gamma\, \mathbf{1}_{[\,x_0, x_0+n^{-1/3}c)}(z),\ z \ge 0,$$

where $c_h = 4g(0)^2/c_0^2$ since for $h^\gamma_{n,(F,x)} \in \mathcal{H}^\gamma_n$ we have

$$\begin{aligned}
& n^{1/6} g(z-x_0) \left| \frac{1}{h_F(z)} - \frac{1}{h_0(z)} \right| \mathbf{1}_{[\,x_0, x_0+n^{-1/3}x)}(z) \\
\leq \quad & n^{1/6} g(0) \frac{|h_0(z) - h_F(z)|}{c_0(c_0/2)} \mathbf{1}_{[\,x_0, x_0+n^{-1/3}x)}(z) \\
\leq \quad & n^{1/6} \frac{2g(0)^2}{c_0(c_0/2)} \, \gamma \, \mathbf{1}_{[\,x_0, x_0+n^{-1/3}c)}(z).
\end{aligned}$$

Note that in our situation

$$\begin{aligned}
\|h^\gamma_{n,(F,x)}\|_{L_2(H_0)} \; & \leq \; \left[\int_{[\,0,\infty)} \left(H^\gamma_n(z) \right)^2 dH_0(z) \right]^{1/2} \; \leq \; \sqrt{g(0)} n^{1/6} c_h \, \gamma \left(n^{-1/3} c \right)^{1/2} \\
= \; & \frac{4g(0)^{5/2} c^{1/2}}{c_0^2} \, \gamma \; =: \kappa_\gamma .
\end{aligned}$$

Note also that by Lemma 5.3.4 (below) which states that the δ-entropy of $\mathcal{H}^\gamma_n$ is of order δ^{-1}, we conclude

$$J_{[\,]}(\delta, \mathcal{H}^\gamma_n, L_2(H_0)) \; \to \; 0 \tag{5.3.4}$$

as $\delta \downarrow 0$ since, for a constant $A > 0$,

$$\int_0^\delta \sqrt{H_{[\,]}(\tau, \mathcal{H}^\gamma_n, L_2(H_0))} \, d\tau \; = \; \int_0^\delta \sqrt{A} \tau^{-1/2} \, d\tau \; \leq \; \sqrt{A\delta}.$$

Now fix $\sigma > 0$ and let $\mu > 0$. Choose $0 < \gamma < c_0/(4g(0))$ such that κ_γ is small enough to guarantee

$$J_{[\,]}(\kappa_\gamma, \mathcal{H}^\gamma_n, L_2(H_0)) \quad \leq \quad \sigma\mu$$

which is possible by (5.3.4). Now choose $n_0 \in \mathbb{N}$ such that $n_0 > ((c_h\gamma)/a(\kappa_\gamma))^3$. Then for all $n \geq n_0$ we have

$$\mathbb{P}\left(\sup_{\mathcal{H}^\gamma_n} \left| \sqrt{n} \int h^\gamma_{n,(F,x)}(z) \, d(H_n - H_0)(z) \right| > \sigma \right) \quad \leq \quad \frac{1}{\sigma} \cdot \sigma\mu + 0 \; = \; \mu$$

by (5.3.3) and the choices of γ and n_0. □

LEMMA 5.3.2 ($L_2(H_0)$-ENTROPY OF A SCALED FUNCTION CLASS).
Let $x_0 > 0$ *and* $m > 0$ *such that* $h_0(z) \geq c_0 > 0$ *for a constant* c_0 *and for all* $z \in [\,x_0, x_0 + m\,]$. *For* $K > 0$ *let*

$$\Phi \quad = \quad \{\varphi : [\,x_0, x_0 + m\,] \to [\,0, K\,]\}\,.$$

Define the scaled class

$$\Phi^s \quad = \quad \Big\{\varphi^s : [\,x_0, x_0 + b\,] \to [\,0, aK\,],\ \varphi^s(y) = a\varphi\Big(x_0 + \frac{m}{b}(y - x_0)\Big)\Big\}$$

for some $0 < b \leq m$ *and* $a > 0$. *Then*

$$H_{[\,]}(\delta, \Phi^s, L_2(H_0)) \quad \leq \quad H_{[\,]}(1/a \cdot \sqrt{(c_0 m)/(g(0)b)}\,\delta, \Phi, L_2(H_0)).$$

PROOF.
Let $\delta > 0$ and $\big\{\big[\,d_i^l \mathbf{1}_{[\,x_0, x_0+m\,]}, d_i^u \mathbf{1}_{[\,x_0, x_0+m\,]}\,\big] : i = 1, \ldots, k\big\}$ be a set of δ-brackets for Φ with respect to $L_2(H_0)$ and with $k = \exp H_{[\,]}(\delta, \Phi, L_2(H_0))$.

Then define, for $i = 1, \ldots, k$,

$$\begin{aligned} d_i^{s,l}(z) \quad &= \quad a d_i^l(x_0 + mb^{-1}(z - x_0))\mathbf{1}_{[\,x_0, x_0+m\,]}(x_0 + mb^{-1}(z - x_0)) \\ &= \quad a d_i^l(x_0 + mb^{-1}(z - x_0))\mathbf{1}_{[\,x_0, x_0+b\,]}(z) \end{aligned}$$

and

$$d_i^{s,u}(z) \quad = \quad a d_i^u(x_0 + mb^{-1}(z - x_0))\mathbf{1}_{[\,x_0, x_0+b\,]}(z)$$

with

$$\begin{aligned} &\|d_i^{s,u} - d_i^{s,l}\|^2_{L_2(H_0)} \\ &= \quad a^2 \int_{x_0}^{x_0+b} \big[d_i^u(x_0 + mb^{-1}(z - x_0)) - d_i^l(x_0 + mb^{-1}(z - x_0))\big]^2 h_0(z)\,dz \\ &= \quad a^2 \int_{x_0}^{x_0+m} \big[d_i^u(u) - d_i^l(u)\big]^2 h_0(x_0 + mb^{-1}(u - x_0))\frac{b}{m}\,du \\ &= \quad a^2 \int_{x_0}^{x_0+m} \big[d_i^u(u) - d_i^l(u)\big]^2 \frac{h_0(x_0 + mb^{-1}(u - x_0))}{h_0(u)}\frac{b}{m}\,dH_0(u) \\ &\leq \quad a^2 \frac{g(0)}{c_0}\frac{b}{m} \cdot \delta^2. \end{aligned}$$

Obviously the set $\Big\{\Big[\,d_i^{s,l}\mathbf{1}_{[\,x_0, x_0+b\,]}, d_i^{s,u}\mathbf{1}_{[\,x_0, x_0+b\,]}\,\Big] : i = 1, \ldots, k\Big\}$ contains brackets for

Φ^s and therefore

$$H_{[\,]}(a\sqrt{g(0)/c_0}\sqrt{b/m}\,\delta, \Phi^s, L_2(H_0)) \quad \leq \quad \log k \;=\; H_{[\,]}(\delta, \Phi, L_2(H_0)). \quad \square$$

Lemma 5.3.3.
Assume the same setup as in Lemma 5.3.1 and let $\gamma < c_0/(4g(0))$ for $c_0 > 0$ being a lower bound of h_0 on $[\,x_0, x_0 + S_g\,]$. Then we have for a constant $A > 0$ (independent of γ and n)

$$H_{[\,]}\left(\delta, \left\{ n^{1/6}\frac{1}{h_F}\mathbf{1}_{[\,x_0, x_0+n^{-1/3}c)} : F \in \mathcal{I}^\gamma \right\}, L_2(H_0)\right) \quad \leq \quad A\frac{1}{\delta}$$

for all $\delta > 0$ if n is chosen such that $[\,x_0, x_0 + n^{-1/3}c) \subset [\,x_0, x_0 + S_g\,]$.

Proof.
The proof is an application of Lemma 5.3.2 that relates the entropy of a scaled function class to the entropy of its non-scaled counterpart. We interpret the function

$$z \quad \mapsto \quad n^{1/6}\frac{1}{h_F(z)}\mathbf{1}_{[\,x_0, x_0+n^{-1/3}c\,]}(z)$$

as the scaled version of

$$z \quad \mapsto \quad \frac{1}{h_F(x_0 + n^{-1/3}c(z - x_0)/S_g)}\mathbf{1}_{[\,x_0, x_0+S_g\,]}(z)$$

in terms of Lemma 5.3.2 with $a = n^{1/6}, b = n^{-1/3}c$ and $m = S_g$.

Let $\delta > 0$. Every h_F with $F \in \mathcal{I}^\gamma$ is a function of bounded variation (since it already holds for $\mathcal{H}$, see Section 3.1) that stays away from zero on $[\,x_0, x_0 + S_g\,]$. Hence by Lemma C.1.1, which states the entropy for such a function class, together with the result of Lemma C.2.1 using $h_F(z) \geq c_0/2$ for all $z \in [\,x_0, x_0 + S_g\,]$ (independent of γ since $\gamma \leq c_0/(4g(0))$), we know that there is an $A_1 > 0$ with

$$\begin{aligned}
&H_{[\,]}\left(\delta, \left\{ \frac{1}{h_F(x_0 + n^{-1/3}c(\,\cdot\, - x_0)/S_g)}\mathbf{1}_{[\,x_0, x_0+S_g\,]}(\,\cdot\,) : F \in \mathcal{I}^\gamma \right\}, L_2(H_0)\right) \\
&\quad \leq \quad A_1\delta^{-1}.
\end{aligned}$$

So we conclude by Lemma 5.3.2, for some constant $A > 0$,

$$\begin{aligned}
&H_{[\,]}\left(\delta, \left\{ n^{1/6}\frac{1}{h_F}\mathbf{1}_{[\,x_0, x_0+n^{-1/3}c\,]} : F \in \mathcal{I}^\gamma \right\}, L_2(H_0)\right) \\
&\quad \leq \quad n^{1/6}\sqrt{\frac{g(0)}{c_0}}\sqrt{\frac{n^{-1/3}c}{S_g}}A_1\delta^{-1} \;=\; A\delta^{-1}. \quad \square
\end{aligned}$$

Lemma 5.3.4 ($L_2(H_0)$-Entropy of $\mathcal{H}_n^\gamma$).
Assume the same set up as in Lemma 5.3.1 and let $\mathcal{H}_n^\gamma$ as in (5.3.1). Let $\gamma \le c_0/(4g(0))$ for $c_0 > 0$ with $h_0(z) \ge c_0$ for all $z \in [\,x_0, x_0 + S_g\,]$. Then there is a constant $A > 0$ such that for all $\delta > 0$

$$H_{[\,]}\big(\delta, \mathcal{H}_n^\gamma, L_2(H_0)\big) \quad \le \quad A\delta^{-1}.$$

Proof.
Let $\delta > 0$ and $\gamma \le c_0/(4g(0))$. Recall that the envelope functions H_n^γ are given by $n^{1/6} c_h \gamma \mathbf{1}_{[\,x_0, x_0+n^{-1/3}c)}$ for $c_h = 4g(0)^2/c_0^2$.

Instead of deriving the entropy of $\mathcal{H}_n^\gamma$ directly, we first consider the class $\mathcal{H}_n^\gamma \oplus \{H_n\} = \mathcal{H}_n^\gamma \oplus \{n^{1/6} c_h \gamma \mathbf{1}_{[x_0, x_0+n^{-1/3}c]}\}$, where $\oplus$ is understood as a pointwise operation on sets (see also Lemma C.2.2) and use the fact that adding a fixed function does not change the entropy of the original class. The advantage is that $\mathcal{H}_n^\gamma \oplus \{H_n\}$ is a class of nonnegative functions so that without loss of generality the same can be assumed for its brackets.
We start by adding the fixed function $-n^{1/6} h_0^{-1} \mathbf{1}_{[\,x_0, x_0+n^{-1/3}c\,]} + H_n^\gamma$ to the class that is studied in Lemma 5.3.3, so that we can find an $A_1 > 0$ such that

$$H_{[\,]}\left(\delta, \left\{n^{1/6}\left(\left(\frac{1}{h_F} - \frac{1}{h_0}\right) + c_h\gamma\right)\mathbf{1}_{[\,x_0, x_0+n^{-1/3}c\,]} : F \in \mathcal{I}^\gamma\right\}, L_2(H_0)\right) \le A_1\delta^{-1}. \tag{5.3.5}$$

Let $n_I = \exp H_{[\,]}(\delta, \{n^{1/6}\left(\left(h_F^{-1} - h_0^{-1}\right) + c_h\gamma\right)\mathbf{1}_{[\,x_0, x_0+n^{-1/3}c\,]} : F \in \mathcal{I}^\gamma\}, L_2(H_0))$ with brackets $\{[c_i^l, c_i^u] : i = 1, \ldots, n_I\}$ defined on $[\,x_0, x_0{+}n^{-1/3}c]$. Then, for all $i = 1, \ldots, n_I$, $0 \le c_i^l \le 2n^{1/6} c_h \gamma \mathbf{1}_{[\,x_0, x_0+n^{-1/3}c\,]}$. And from (5.3.5) we know that $\log(n_I) \le A_1\delta^{-1}$.

Let $n_K = \exp H_{[\,]}(\delta, \{n^{1/6} c_h \gamma\left(1 - \mathbf{1}_{[\,x_0, x_0+n^{-1/3}x)}\right) : x \in [\,0, c]\}, L_2(H_0))$ with brackets $\{[\,b_k^l, b_k^u\,] : k = 1, \ldots, n_K\}$. Then by Lemma 5.3.2 we have $\log(n_K) \le A_2\delta^{-1}$ with a constant $A_2 > 0$.

Now define the following grid on $[\,0, c\,]$:

$$t_0 = 0,\ t_1 = \delta^2,\ t_2 = 2\delta^2,\ \ldots,\ t_{n_J} = n_J\delta^2$$

with $t_{n_{J-1}} \le c$ and $t_{n_J} > c$. Note that $n_J \le A_3\delta^{-2}$ for some positive constant A_3. Define the functions

$$d_{ijk}^l(z) \quad = \quad c_i^l(z)\mathbf{1}_{[\,x_0, x_0+n^{-1/3}t_{j-1})}(z) + b_k^l(z)$$

and

$$d_{ij}^u(z) \quad = \quad c_i^u(z)\mathbf{1}_{[\,x_0, x_0+n^{-1/3}t_j)}(z) + b_k^u(z)$$

for every $(i,j,k) \in \{1,\ldots,n_I\} \times \{1,\ldots,n_J\} \times \{1,\ldots,n_K\}$.

Now let $h^{\gamma}_{n,(F,x)} + n^{1/6} c_h \gamma \mathbf{1}_{[x_0, x_0 + n^{-1/3} c]} \in \mathcal{H}^{\gamma}_n \oplus \{H^{\gamma}_n\}$. Note that

$$
\begin{aligned}
& n^{1/6} \left(\frac{1}{h_F} - \frac{1}{h_0} \right) \mathbf{1}_{[\,x_0, x_0 + n^{-1/3} x)} + n^{1/6} c_h\, \gamma \mathbf{1}_{[\,x_0, x_0 + n^{-1/3} c\,]} \\
&= \; n^{1/6} \left[\left(\frac{1}{h_F} - \frac{1}{h_0} \right) + c_h\, \gamma \right] \mathbf{1}_{[\,x_0, x_0 + n^{-1/3} c\,]} \mathbf{1}_{[\,x_0, x_0 + n^{-1/3} x)} \\
&\qquad\qquad\qquad\qquad\qquad + n^{1/6} c_h\, \gamma \mathbf{1}_{[\,x_0 + n^{-1/3} x, x_0 + n^{-1/3} c\,]} \\
&= \; n^{1/6} \left[\left(\frac{1}{h_F} - \frac{1}{h_0} \right) + c_h\, \gamma \right] \mathbf{1}_{[\,x_0, x_0 + n^{-1/3} c\,]} \mathbf{1}_{[\,x_0, x_0 + n^{-1/3} x)} \\
&\qquad\qquad\qquad\qquad\qquad + n^{1/6} c_h\, \gamma (1 - \mathbf{1}_{[\,x_0, x_0 + n^{-1/3} x)}).
\end{aligned}
$$

So there exits a bracket $\left[d^l_{ijk}, d^u_{ijk} \right]$ such that

$$
d^l_{ijk} \;\; \leq \;\; h^{\gamma}_{n,(F,x)} + n^{1/6} c_h\, \gamma \mathbf{1}_{[x_0, x_0 + n^{-1/3} c]} \;\leq\; d^u_{ijk}
$$

where we use the positivity of the functions c^l_i. Furthermore,

$$
\begin{aligned}
\| d^u_{ijk} - d^l_{ijk} \|_{L_2(H_0)} \;&=\; \| c^u_i \mathbf{1}_{[\,x_0, x_0 + n^{-1/3} t_j)} + b^u_k - c^l_i \mathbf{1}_{[\,x_0, x_0 + n^{-1/3} t_{j-1})} - b^l_k \|_{L_2(H_0)} \\
\leq \;\; & \| \left(c^u_i - c^l_i \right) \mathbf{1}_{[\,x_0, x_0 + n^{-1/3} t_j)} \|_{L_2(H_0)} + \| c^l_i \mathbf{1}_{[\,x_0 + n^{-1/3} t_{j-1}, x_0 + n^{-1/3} t_j)} \|_{L_2(H_0)} \\
& \qquad\qquad\qquad + \| b^u_k - b^l_k \|_{L_2(H_0)} \\
\leq \;\; & \delta + 2 c_h\, \gamma n^{1/6} \left(n^{-1/3} (t_i - t_{i-1}) \right)^{1/2} + \delta \;\leq\; (2 + 2 c_h\, \gamma) \delta \\
\leq \;\; & \left(2 + 2 c_h \frac{c_0}{4 g(0)} \right) \delta \;=\; \tilde{c} \delta
\end{aligned}
$$

for $\tilde{c} = 2 + 2 c_h c_0 / (4 g(0)) > 0$.

Hence

$$
\begin{aligned}
H_{[\,]}(\tilde{c}\delta, \mathcal{H}^{\gamma}_n, L_2(H_0)) \;&=\; H_{[\,]}(\tilde{c}\delta, \mathcal{H}^{\gamma}_n \oplus \{H^{\gamma}_n\}, L_2(H_0)) \\
\leq \;\; & \log n_I n_J n_K \;\leq\; (A_1 \delta^{-1} + A_2 \delta^{-1} + \log(A_3 \delta^{-2}) \;=\; A \delta^{-1}
\end{aligned}
$$

for some $A > 0$, independent of γ. $\square$

LEMMA 5.3.5 ($\tilde{R}_{n,4}$).
Let $x_0 \in (0, S_0)$ as in Assumption 5.1.1, $\lambda > 0$ as in the Remark thereafter such that in addition

$$\sup_{z\in[\,x_0-\lambda,x_0+\lambda\,]} \int_{[\,0,z\,]} \big(g(z-y)-g(0)\big)\,d(\hat{F}_n - F_0)(y)\,dy \quad = \quad O_p(n^{-1/2}). \qquad (5.3.6)$$

Then

$$\frac{1}{t-s}\left|\int_{[\,s,t)} \frac{g(z-s)}{\hat{h}_n(z)} \int_{[\,0,z)} g'(z-y)\left(\hat{F}_n(y)-F_0(y)\right)\,dy\,dz\right| \quad = \quad O_p(n^{-1/2})$$

uniformly for $x_0 - \lambda \le s < t \le x_0 + \lambda$.

PROOF.
The proof is analogous to the one of Lemma 4.4.8. Since the additional assumption (5.3.6) can be written as

$$\int_{[\,0,z\,]} \big(g(z-y)-g(0)\big)\,d(\hat{F}_n - F_0)(y)\,dy \quad = \quad \int_{[\,0,z\,]} g'(z-y)\left(\hat{F}_n(y)-F_0(y)\right)\,dy,$$

using integration by parts, we can replace the far right side in (4.4.7) by $c_2 n^{-1/2}$ for some positive constant c_2, which implies the claimed rate. □

LEMMA 5.3.6 ($\tilde{R}_{n,6}$).
Under Assumption 5.1.1, $\sup_{x\in[\,-c,c\,]} n^{2/3}|\tilde{R}_{n,6}(x)| = o_p(1)$, $n \to \infty$, for all $c > 0$.

PROOF.
Let $c > 0$ and $\eta > 0$. According to Assumption 5.1.1 one can find a $\lambda > 0$ such that $|f_0(x_0) - f_0(x)| < 2\eta/c^2$ whenever $|x - x_0| < \lambda$. Choose n sufficiently large such that $[\,x_0, x_0 + n^{-1/3}c\,] \subset [\,x_0, x_0 + \lambda\,]$. Then, since $\xi_z \in (x_0, x_0 + n^{-1/3}x) \subset (x_0, x_0 + n^{-1/3}c) \subset [\,x_0, x_0 + \lambda\,]$,

$$\begin{aligned}
&\frac{h_0(x_0)}{g(0)^2} \sup_{x\in[\,0,c\,]} n^{2/3}\left|\tilde{R}_{n,6}(x)\right| \\
&= \sup_{x\in[\,0,c\,]}\left| n^{2/3}\int_{[\,x_0,x_0+n^{-1/3}x)} \big(F_0(x_0)-F_0(z)\big)\,dz + \frac{f_0(x_0)}{2}x^2\right| \\
&= \sup_{x\in[\,0,c\,]}\left| n^{2/3}\int_{[\,x_0,x_0+n^{-1/3}x)} [-f_0(\xi_z)+f_0(x_0)]\,(z-x_0)\,dz\right| \\
&\le \sup_{x\in[\,0,c\,]} 2\eta/c^2 \left| n^{2/3}\left[-\frac{1}{2}(z-x_0)^2\Big|_{x_0}^{x_0+n^{-1/3}x}\right]\right| \;<\; \eta. \quad \square
\end{aligned}$$

PROOF OF LEMMA 5.1.6.
Apply decomposition (4.2.3) with $t = x_0$ and $s = \tau^+$ so that

$$n^{2/3}\left(\hat{C}_n(x_0) - \hat{C}_n(\tau^+)\right) = n^{2/3}\int_{[\,x_0,\tau^+)} \frac{g(z-x_0)}{h_0(z)}\,d(H_n - H_0)(z)$$

$$-n^{2/3}\frac{g(0)^2}{h_0(x_0)}\int_{[\,x_0,\tau^+)}\left(\hat{F}_n(z) - F_0(z)\right)\,dz + n^{2/3}\sum_{i=1}^{5} R_{n,i}(x_0,\tau^+). \qquad (5.3.7)$$

Note that by Lemma 4.3.5 there exists a $T > 0$ such that the event $\{\tau^+ - x_0 \le T n^{-1/3}\}$ holds with arbitrarily high probability for sufficiently large n. Also, choose n_0 such that $n_0^{-1/3}T \le \lambda$ for $\lambda > 0$ as in the Remark after Assumption 5.1.1.

The first term of the right of (5.3.7) is bounded with arbitrarily high probability by

$$\sup_{t\in[\,0,T\,]}\left|n^{1/2}\int_{[\,0,\infty)} h_{n,t}(z)\,d(H_n - H_0)(z)\right|$$

for $h_{n,t}$ an element of

$$\mathcal{H}_n^T = \left\{h_{n,t}\,|\,h_{n,t}(z) = n^{1/6}\frac{g(z-x_0)}{h_0(z)}\mathbf{1}_{[\,x_0,x_0+n^{-1/3}t)}(z) : t\in[\,0,T\,], z\ge 0\right\},$$

which contains h_{n,τ^+} with arbitrarily high probability for n sufficiently large. Using the same argument as in (5.3.3) we write, for $n > n_0$,

$$\mathbb{P}\left(\sup_{\mathcal{H}_n^T}\left|\sqrt{n}\int_{[\,0,\infty)} h_{n,t}(z)\,d(H_n - H_0)(z)\right| > T\right)$$

$$\le \frac{1}{T}\sqrt{n}\int H_n^T(z)\mathbf{1}_{H_n^T(z)>\sqrt{n}\alpha(\kappa)}(z)\,dz + \frac{1}{T}J_{[\,]}\left(\kappa,\mathcal{H}_n^T, L_2(H_0)\right) \qquad (5.3.8)$$

where also here H_n^T denotes an envelope function of $\mathcal{H}_n^T$, where $\kappa \ge \|h_{n,t}\|_{L_2(H_0)}$ for all $h_{n,t}\in\mathcal{H}_n^T$, where $a(\kappa) = \kappa/\sqrt{1 + H_{[\,]}(\kappa,\mathcal{H}_n^T, L_2(H_0))}$ and where $J_{[\,]}(\delta,\mathcal{H}_n^T, L_2(H_0)) = \int_0^\delta \sqrt{H_{[\,]}(\tau,\mathcal{H}_n^T, L_2(H_0))}\,d\tau$.

Let the envelope of $\mathcal{H}_n^T$ be $H_n^T(z) = n^{1/6}g(z-x_0)/h_0(z)\mathbf{1}_{[\,x_0,x_0+n^{-1/3}T)}(z)$ and $\kappa = g(0)^2/c_0\sqrt{T}$ where c_0 is the lower bound of h_0 on $[\,x_0, x_0+\lambda]$. Define for $\delta > 0$ a grid $0 = t_0 < t_1, \cdots, t_{K-1} \le T < t_K$ with mash width δ^2. Then

$$\left\{\left[n^{1/6}\frac{g(\,\cdot\,-x_0)}{h_0(\,\cdot\,)}\mathbf{1}_{[\,x_0,x_0+n^{-1/3}t_{i-1})}, n^{1/6}\frac{g(\,\cdot\,-x_0)}{h_0(\,\cdot\,)}\mathbf{1}_{[\,x_0,x_0+n^{-1/3}t_i)}\right] : i = 1,\ldots,K\right\}$$

acts as δ-brackets for $\mathcal{H}_n^T$. Thus, $H_{[\,]}(\delta,\mathcal{H}_n^T, L_2(H_0)) \le A^T\delta^{-1}$ for some constant $A^T > 0$, and hence $J_{[\,]}(\delta, \mathcal{H}_n^T, L_2(H_0)) \leq A_1^T \delta$ for a constant $A_1^T > 0$.

Now let $\eta > 0$. Choose $n_1 > \max\{n_0, (g(0)/c_0 a(\kappa))^3\}$. Then

$$\mathbb{P}\left(\sup_{\mathcal{H}_n^T}\left|\sqrt{n}\int_{[\,0,\infty)} h_{n,t}(z)\, d(H_n - H_0)(z)\right| > T\right) \;\leq\; \eta$$

for all $n \geq n_1$ by choosing T sufficiently large due to (5.3.8).

With probability tending to one, we have

$$\left|n^{2/3}\frac{g(0)^2}{h_0(x_0)}\int_{[\,x_0,\tau^+)}\left(\hat{F}_n(z) - F_0(z)\right)\, dz\right|$$

$$\leq \quad n^{2/3}\frac{g(0)^2}{h_0(x_0)}\sup_{[\,x_0, x_0+Tn^{-1/3}\,]}\left|\hat{F}_n(z) - F_0(z)\right|(\tau^+ - x_0)$$

which is tight by applying the local rate results for $\hat{F}_n$ derived in Lemma 4.3.3 and the rate of successive points of jump of $\hat{F}_n$ (Lemma 4.3.5).

Note also that for n sufficiently large

$$n^{2/3}\sum_{i=1}^{5} R_{n,i}(x_0, \tau^+) \quad = \quad O_p(1)$$

due to Lemma 4.2.2 which completes this proof. □

LEMMA 5.3.7.
Assume the same setup as in Section 5.2. Then

$$K_t(\hat{F}_n) - K_t(F_0) \quad = \quad -\int_{[\,0,2\,]} b_{t,\hat{F}_n}(z)\, dH_0(z).$$

PROOF.
Using (5.2.20) we obtain

$$\int_{[\,0,\infty)} b_{t,\hat{F}_n}(z)\, dH_0(z) \;=\; \int_{z\in[\,0,2\,]} b_{t,\hat{F}_n}(z)\int_{x\in[\,0,z\,]} g(z-x)\, dF_0(x)\, dz$$

$$= \quad \int_{x\in[\,0,2\,]}\int_{z\in[\,x,x+1\,]} b_{t,\hat{F}_n}(z) g(z-x)\, dz\, dF_0(x)$$

$$= \quad \int_{x\in[\,0,2\,]}\left[k(x) - \int k(u)\, d\hat{F}_n(u)\right] dF_0(x) \;=\; K_t(F_0) - K_t(\hat{F}_n). \quad \square$$

Lemma 5.3.8.
Suppose $\bar{A}$ is a piecewise constant function such that the locations of its point of jump coincide with the elements of $\mathcal{T}_n$ with $\bar{A}(x) = 0$ for all $x \in (-\infty, \tau_1) \cup [\,\tau_m, \infty)$. Then

$$\int_{x\in[\,0,\infty)} \int_{z\in[\,x,\infty)} \frac{g(z-x)}{\hat{h}_n(z)}\, dH_n(z)\, d\bar{A}(x) \quad = \quad 0.$$

Proof.
By the Fenchel optimality conditions the inner integral equals 1 whenever x is a point of jump of $\hat{F}_n$. The piecewise constant structure of $\bar{A}$ then yields

$$\int_{[\,0,\infty)} \int_{[\,x,\infty)} \frac{g(z-x)}{\hat{h}_n(z)}\, dH_n(z)\, d\bar{A}(x) \quad = \quad \bar{A}(\tau_1-) - \bar{A}(\tau_m)$$

which equals zero by the definition of $\bar{A}$. □

Lemma 5.3.9.
Let $b_{t,\hat{F}_n}$ and $\bar{b}_{t,\hat{F}_n}$ as in Section 5.2. Then

$$\int \Big(\bar{b}_{t,\hat{F}_n}(z) - b_{t,\hat{F}_n}(z)\Big)\, \hat{h}_n(z)\, dz \quad = \quad 0.$$

Proof.
Direct calculations yield

$$\begin{aligned}
\int \Big(b_{t,\hat{F}_n}(z) - \bar{b}_{t,\hat{F}_n}(z)\Big)\, \hat{h}_n(z)\, dz \;&=\; \int_{Z_{(1)}}^{1+\hat{S}_n} \Big(b_{t,\hat{F}_n}(z) - \bar{b}_{t,\hat{F}_n}(z)\Big)\, \hat{h}_n(z)\, dz \\
=\quad & \int_{z=0}^{1+\hat{S}_n} \int_{x\in[\,Z_{(1)},z\,]} g(z-x)\, d(A_{t,\hat{F}_n} - \bar{A}_{t,\hat{F}_n})(x)\, dz \\
=\quad & \int_{x\in[\,Z_{(1)},\hat{S}_n\,]} \int_{z=x}^{x+1} g(z-x)\, dz\, d(A_{t,\hat{F}_n} - \bar{A}_{t,\hat{F}_n})(x) \\
=\quad & \int_{x\in[\,Z_{(1)},\hat{S}_n\,]} dA_{t,\hat{F}_n}(x) - \bar{A}_{t,\hat{F}_n}(\hat{S}_n) \;=\; 0
\end{aligned}$$

where the last equality is due to (5.2.15). □