4 | Rigorous validation of a Hopf bifurcation in the Kuramoto–Sivashinsky PDE

4.1 Introduction

The goal of this paper is to give a rigorous validation of a Hopf bifurcation found numerically in the Kuramoto-Sivashinsky partial differential equation (PDE) in one spatial dimension with $2\pi$-periodic boundary conditions in space:

\[
\begin{aligned}
\partial_t u &= -\partial_x^2 u - \gamma \partial_x^4 u + \partial_x (u^2), \\
u(t, x) &= u(t, x + 2\pi).
\end{aligned}
\] (4.1)

Its appealing qualities are its relative simplicity and the presence of a low order non-linear term. Despite its simplicity, it exhibits complex dynamics, and serves as a model for weak turbulence in laminar flows and more generally as a paradigm for studying spatiotemporal chaos. We refer to [56], [31] and [57] for more background material on the Kuramoto-Sivashinsky equation.

The Kuramoto–Sivashinsky equation has been well studied from the perspective of validated numerics. Early results include [80], where stationary solutions of (4.1) are found and validated, while the recent paper [76] uses rigorous numerics to prove the existence of chaos in the Kuramoto–Sivashinsky equation. Furthermore, the bifurcation diagram of stationary solutions of Kuramoto-Sivashinsky and their stability was studied in [1], see also [82] for the pitchfork bifurcation problem. We refer to [19], [20] and [79, 2] for
validated continuation of periodic orbits in Kuramoto–Sivashinsky. Here we complement these results by an analysis of the Hopf bifurcation problem. Indeed, the periodic solutions proven in [19, 20, 82] bifurcate from an inhomogeneous stationary state, but a mathematically rigorous analysis of this Hopf bifurcation problem has not previously been performed. To tackle this, we extend out previous work on ODEs [66] to the PDE setting. The functional analytic setup builds on [19] and [20] and our blow up technique is similar in spirit to [1] and [82] but allows for bifurcations from nonhomogeneous stationary states.

In this paper we prove the existence of a Hopf bifurcation from a nonhomogeneous stationary solution to (4.1) by a posteriori validation of numerical computations. This result is achieved by a blow up procedure in the neighborhood of the Hopf bifurcation. More precisely, we consider a time $L$-periodic, space $2\pi$-periodic solution $u(t, x)$ to (4.1). We apply a time rescaling $\tau = \frac{2\pi t}{L}$ such that $u(\tau, x)$ is time $2\pi$-periodic. The rescaled equation is

$$
\begin{align*}
\dot{u} &= -\lambda_1 \partial_x^4 u - \lambda_1 \lambda_2 \partial_x^2 u + \lambda_1 \partial_x (u^2), \\
\dot{u} &= u(\tau, x + 2\pi).
\end{align*}
$$

(4.2)

with $\lambda_1 = \frac{L}{2\pi}$ and $\lambda_2 = \gamma$.

If a time-dependent solution $u(\tau, x)$ is close enough to the Hopf bifurcation, it is natural to rewrite $u$ as the sum of a stationary solution $y(x)$ to (4.2) and a time-dependent perturbation $az(\tau, x)$, where $z$ is of order 1 and $a \in \mathbb{R}$ is the amplitude of the perturbation. The precise statements can be found in Section 4.2.3. Following this rewriting, when we validate a branch of solutions $x = (\lambda_1, \lambda_2, a, y, z)$ and the amplitude $a$ changes sign along this branch, then at $a = 0$ a Hopf bifurcation occurs: the periodic solutions for $a \neq 0$ bifurcate from the stationary state $y$ at $a = 0$. This result is presented in Section 4.9. The main challenge in transforming the ODE results in [66] to the PDE setting of the current paper are two-fold. First, the two-dimensional Fourier series lead to diagonally dominant operators that depend on the a priori unknown periodic (in time). This requires additional, somewhat technical, estimates. Second, the nonlinearity includes a derivative, which can be controlled because it is lower order, but it requires some care. In fact, it is remarkable how congruous the ODE and PDE problems are in the chosen functional analytic framework on Fourier space. We remark here that the Hopf bifurcation thus proven might be degenerate, since we do not check the normal form in the neighborhood of the bifurcation nor do we study the behavior of the eigenvalues of the problem. The non-degeneracy could be proven by applying the reformulation of a Hopf bifurcation into a saddle node bifurcation, as already introduced in [66] for ODEs and proving
the existence of a saddle node in the new system. We are confident such approach would work in the PDE setting as well.

The blow up approach presented is extremely general in its simplicity. The extension of the method to a broader class of PDEs is relatively straightforward, although beyond the scope of this article. In particular, it seems probable that the bounds developed in this paper can be applied with minor modifications to many other parabolic PDEs with periodic (or Neuman) boundary conditions.

4.2 The Kuramoto-Sivashinsky equation and the Hopf bifurcation system

4.2.1 The Kuramoto-Sivashinsky equation

In Section 4.1, we rescaled the Kuramoto-Sivashinsky equation to

\[
\partial_t v = -\lambda_1 \lambda_2 \partial_x^4 v - \lambda_1 \partial_x^2 v + 2\lambda_1 \partial_x (v^2),
\]

(4.3)

so that we can study solutions which are $2\pi$-periodic in space and time.

To write $v$ in terms of its Fourier expansion in $t$ and $x$ we introduce the two-dimensional bi-infinite sequence $u = (u_{jk})_{j, k \in \mathbb{Z}}$, with $u_{jk} \in \mathbb{C}$, such that

\[
v(t, x) = \sum_{j, k \in \mathbb{Z}} u_{jk} e^{i(jt+kx)}.
\]

(4.4)

We remark that we will coherently use the letter $j$ for time related indices and $k$ for space related indices.

We introduce the notation

\[
\mathcal{F} v = u
\]

for the two-dimensional Fourier transform, with $u$ defined above.

There are some standard properties of Fourier sequences that will be used extensively in the following. The convolution of two two-dimensional bi-infinite sequences $u$ and $w$ is defined by

\[
(u * w)_{jk} = \sum_{j_1+j_2=j} \sum_{k_1+k_2=k} u_{j_1k_1} w_{j_2k_2}.
\]

Furthermore it holds that

\[
u * w = \mathcal{F}^{-1} (u \cdot \mathcal{F}^{-1}(w))
\]

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where $F^{-1}$ is the inverse Fourier transform and the multiplication $\cdot$ is considered pointwise in $t$ and $x$. The derivative operators satisfy

$$F(\partial_t v(t, x)) = (i j u_{jk})_{j, k \in \mathbb{Z}} \quad \text{and} \quad F(\partial_x v(t, x)) = (i k u_{jk})_{j, k \in \mathbb{Z}}.$$ 

We denote the multiplication of a two-dimensional bi-infinite sequence by its index in the time or space direction by the operator $J$ or $K$, respectively, defined as

$$J : u \mapsto (j u_{jk})_{j, k \in \mathbb{Z}} \quad \text{(4.5)}$$

and

$$K : u \mapsto (k u_{jk})_{j, k \in \mathbb{Z}}. \quad \text{(4.6)}$$

Thus we are able to write $F(\partial_t v) = iJF(v)$ and $F(\partial_x v) = iKF(v)$.

Using the notation just introduced, we can rewrite Equation (4.3) in the space of Fourier sequences as

$$iJ u = -\lambda_1 \lambda_2 K^4 u + \lambda_1 K^2 u + i \lambda_1 K (u \ast u) \quad \text{(4.7)}$$

and we introduce the zero finding problem

$$f(\lambda_1, \lambda_2, u) \overset{\text{def}}{=} iJu + \lambda_1 \lambda_2 K^4 u - \lambda_1 K^2 u - i \lambda_1 K (u \ast u) = 0. \quad \text{(4.8)}$$

**Remark 4.2.1.** In Equation (4.3), the last term is $2uv_x = (v^2)_x$. The first formulation would be rewritten as $2u \ast (iKu)$ in Fourier sequences, while the second one corresponds to $iK(u \ast u)$. They are equivalent, but the second one is chosen in this paper because it makes the notation for derivatives simpler.

**Remark 4.2.2.** Writing explicitely

$$f_{jk}(\lambda_1, \lambda_2, u) = ij u_{jk} + \lambda_1 \lambda_2 k^4 u_{jk} - \lambda_1 k^2 u_{jk} - \lambda_1 ik (u \ast u)_{jk}, \quad j, k \in \mathbb{Z},$$

we notice that $f_{00}(\lambda_1, \lambda_2, u) = 0$.

We remark that, if $u$ is a solution to (4.8), any shift in time or space of $u$ is also a solution. In order to avoid that, we add two phase conditions to our problem of the form

$$\langle iK\tilde{u}, u \rangle = 0 \quad \text{and} \quad \langle iJ\tilde{u}, u \rangle = 0, \quad \text{(4.9)}$$

where $\tilde{u}$ is an approximation of $u$ and

$$\langle u, w \rangle = \sum_{j, k \in \mathbb{Z}} u_{jk} w_{jk},$$
for \( u, w \in \mathbb{C}^2 \). Conditions (4.9) are the Fourier sequence representations of

\[
\int_0^{2\pi} \tilde{v} \tilde{x} dx = 0 \quad \text{and} \quad \int_0^{2\pi} \tilde{v} \tilde{t} dt = 0,
\]

with \( \tilde{v} \) a numerically determined approximation of \( v \). These are standard conditions to fix shifts, [74].

The full problem then becomes

\[
\begin{align*}
  f_{jk}(\lambda_1, \lambda_2, u) &= 0, \quad \text{for all } (j, k) \in \mathbb{Z}^2 \setminus \{(0,0)\}, \\
  \langle iK\tilde{u}, u \rangle &= 0, \\
  \langle i\mathbf{J}\tilde{u}, u \rangle &= 0,
\end{align*}
\]

(4.10)

and one expects a one-dimensional solution branch, due to the two parameters, two scalar equations and considering Remark 4.2.2.

### 4.2.2 The functional spaces \( \ell_{\nu_2}^1 \) and \( L_\nu^1 \)

In this section, we introduce the spaces of one and two-dimensional bi-infinite sequences. Following [5], for one-dimensional bi-infinite sequences we introduce the norm

\[
\|u\|_{\ell_{\nu_2}^1} \overset{\text{def}}{=} \sum_{k \in \mathbb{Z}} \nu_{2}^{|k|} |u_k|,
\]

(4.11)

and the space of one-dimensional exponentially decreasing sequences as

\[
\ell_{\nu_2}^1 = \{u = (u_k)_{k \in \mathbb{Z}}, u_k \in \mathbb{C} | \|u\|_{\ell_{\nu_2}^1} < \infty \}.
\]

The convolution of two sequences \( y \) and \( w \) in \( \ell_{\nu_2}^1 \) is defined as

\[
(y \ast w)_k = \sum_{k_1 + k_2 = k} y_{k_1} w_{k_2}.
\]

The norm and space introduced for bi-infinite sequences are very similar. For a given vector \( \nu = (\nu_1, \nu_2) \), with \( \nu_i \geq 1 \) for \( i = 1, 2 \), we introduce the \( L_\nu^1 \)-norm in the space of bi-infinite sequences

\[
\|u\|_{L_\nu^1} \overset{\text{def}}{=} \sum_{j,k \in \mathbb{Z}} \nu_1^{j} \nu_2^{k} |u_{jk}|,
\]

(4.12)

that is the two-dimensional version of the norm already used in [74]. It resembles closely the norm used in other papers discussing Kuramoto-Sivashinsky, such as [19] and [20]. In particular, the \( L_\nu^1 \)-norm has the similar properties
to the $M$-norm presented in [19]. We define the space of two-dimensional exponentially decreasing sequences as

$$\mathcal{L}_\nu^1 = \{ u = (u_{j,k})_{j,k \in \mathbb{Z}}, u_{j,k} \in \mathbb{C} \mid \| u \|_{\mathcal{L}_\nu^1} < \infty \}. $$

It is then clear that $\mathcal{L}_\nu^1$ is an extension of the $\ell_{\nu_2}^1$ space for two-dimensional sequences where the exponential decrease in the first index is determined by $\nu_1$ and in the second index by $\nu_2$.

The $\mathcal{L}_\nu^1$ space shares a lot of properties with the $\ell_{\nu_2}^1$ space. First of all, $\mathcal{L}_\nu^1$ is a Banach space, and

$$\| u * v \|_{\mathcal{L}_\nu^1} \leq \| u \|_{\mathcal{L}_\nu^1} \| v \|_{\mathcal{L}_\nu^1}. $$

Furthermore, there is a trivial embedding of $\ell_{\nu_2}^1$ in $\mathcal{L}_\nu^1$.

$$E_\ell : \ell_{\nu_2}^1 \rightarrow \mathcal{L}_\nu^1$$

$$y \mapsto E_\ell y : (E_\ell y)_{j,k} = \delta_{j_0} y_k,$$ which satisfies $\| y \|_{\ell_{\nu_2}^1} = \| E_\ell y \|_{\mathcal{L}_\nu^1}$. Thanks to this embedding, we can make sense of the convolution between an element $y$ of $\ell_{\nu_2}^1$ and an element $u$ of $\mathcal{L}_\nu^1$, defining it as

$$(y * u)_{j,k} \overset{\text{def}}{=} (E_\ell y * u)_{j,k} = \sum_{k_1 + k_2 = k} y_{k_1} u_{j,k_2}. $$

### 4.2.3 The Hopf bifurcation

A sequence $y \in \ell_{\nu_2}^1$ is a stationary solution of (4.8) if it satisfies

$$F_1(\lambda_2, y) \overset{\text{def}}{=} \lambda_2 K^4 y - K^2 y - i K(y * y) = 0, \quad (4.13)$$

where the equation is considered in $\ell_{\nu_2}^1$. We remark that $(F_1)_0 = 0$ is trivially satisfied. To avoid shifts in space of the solution to (4.13), as we did for the time-dependent solution in Section 4.2.1, we add the phase condition

$$\langle iK\tilde{y}, y \rangle_\ell \overset{\text{def}}{=} \langle iE_\ell K\tilde{y}, E_\ell y \rangle = \sum_{k \in \mathbb{Z}} ik\tilde{y}_k y_k = 0, \quad (4.14)$$

with $\tilde{y}$ an approximation of $y$.

Similarly to the Hopf bifurcation validation presented in [66], we use a blow up approach in the neighborhood of a Hopf bifurcation. To this end, we rewrite the solution $u \in \mathcal{L}_\nu^1$ to (4.10) in the neighborhood of a Hopf
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bifurcation as the sum of the stationary solution \( y \in \ell^1_{\nu_2} \) to (4.13) and a perturbation \( az \), with \( z \in L^1_{\nu} \) of norm 1 and amplitude \( a \in \mathbb{R} \), that is

\[
\begin{cases}
    u = E\ell y + az, \\
    \|z\|^2_{L^1_{\nu}} = 1,
\end{cases}
\]

(4.15)

where the second equation ensures that \( \|az\|_{L^1_{\nu}} = |a| \), that is, \( a \) is the amplitude of \( az \), and \( z \) does not tend to a singular (stationary) orbit as the Hopf bifurcation point is approached.

However, we are not interested in computing precisely the norm of \( z \); we are just interested in imposing a non-degeneracy condition on \( z \). Therefore we replace the second equation with the easier to compute

\[
\langle J^2\tilde{z}, z \rangle - 1 = 0,
\]

with \( \tilde{z} \) a numerically determined approximation of \( z \).

Replacing \( u \) in (4.10) by \( E\ell y + az \), we have

\[
\begin{cases}
    \langle J^2\tilde{z}, z \rangle = 1, \\
    \langle iK(E\ell \tilde{y} + \tilde{a} \tilde{z}), (E\ell y + az) \rangle = 0, \\
    \langle iJ(E\ell \tilde{y} + \tilde{a} \tilde{z}), (E\ell y + az) \rangle = 0, \\
    iJ(E\ell y + az) + \lambda_1 \lambda_2 K^4(E\ell y + az) \\
    -\lambda_1 K^2(E\ell y + az) - \lambda_1 iK((E\ell y + az) \ast (E\ell y + az)) = 0.
\end{cases}
\]

(4.16)

We add the equations (4.13) and (4.14) and use this information on \( y \) to simplify the terms involving only \( y \) in the last equation. Concerning the phase condition in time, we recall that \( y \) is time-independent, therefore \( Jy = 0 \). Following a division by \( a \), the problem then becomes

\[
\begin{cases}
    \langle J^2\tilde{z}, z \rangle - 1 = 0, \\
    \langle iK\tilde{y}, y \rangle_{\ell} = 0, \\
    \langle iK(E\ell \tilde{y} + \tilde{a} \tilde{z}), z \rangle = 0, \\
    \langle iJ\tilde{z}, z \rangle = 0, \\
    \lambda_2 K^4y - K^2y - iK(y \ast y) = 0, \\
    iJz + \lambda_1 \lambda_2 K^4z - \lambda_1 K^2z - \lambda_1 iK(2E\ell y \ast z + az \ast z) = 0.
\end{cases}
\]

(4.17)

Remark 4.2.3. The equation \( \langle iK(E\ell \tilde{y} + \tilde{a} \tilde{z}), z \rangle = 0 \) is not the only choice for the phase condition on \( z \). Other choices are possible, such as setting \( \langle iK\tilde{z}, z \rangle = 0 \). In particular, the choice made in this paper is unfit in case \( y \) is space-independent, because in that case the equation would be trivially satisfied at the bifurcation point. In the case at hand, \( y \) is space-dependent, hence the stated condition is appropriate.
Let \( \hat{x} = (\hat{\lambda}_1, \hat{\lambda}_2, \hat{a}, \hat{y}, \hat{z}) \) be a numerical approximation of the solution of (4.17), with \( \hat{y} \) and \( \hat{z} \) having just a finite number of non-zero Fourier coefficients. We use the numerical approximation to fix the phase conditions. Then the problem we want to solve is

\[
\begin{align*}
G^\hat{x}(\lambda_1, \lambda_2, a, y, z) &= \begin{pmatrix}
\langle J^2 \hat{z}, z \rangle - 1 \\
\langle iK \hat{y}, y \rangle \\
\langle iK (E \hat{y} + a \hat{z}), z \rangle \\
\langle iJ \hat{z}, z \rangle 
\end{pmatrix} = 0, \\
F(\lambda_1, \lambda_2, a, y, z) &= \begin{pmatrix}
F_1(\lambda_2, y) \\
F_2(\lambda_1, \lambda_2, a, y, z)
\end{pmatrix} = 0.
\end{align*}
\tag{4.18}
\]

Here we also divide the problem into a linear problem \( G^\hat{x} \), where we emphasize the dependency on the numerical approximation, and a non-linear problem \( F \), with \( F_1 \) as defined in (4.13) and

\[
F_2(\lambda_1, \lambda_2, a, y, z) \overset{\text{def}}{=} iJz + \lambda_1 \lambda_2 K^4 z - \lambda_1 K^2 z - \lambda_1 iK(2E_l y * z + a z * z).
\]

We will at times use the notation \((F_1)_k\) and \((F_2)_{jk}\) to indicate the \(k\)-th and \((j,k)\)-th Fourier coefficients of \(F_1(x)\) and \(F_2(x)\) respectively, where the argument will be clear from context.

The set of variables for (4.18) has the form \((\lambda_1, \lambda_2, a, y, z)\) with \(\lambda_1, \lambda_2, a \in \mathbb{C}, y \in \ell^1_{\nu_2} \text{ and } z \in L^1_{\nu}\). The full space of variables is thus

\[
X \overset{\text{def}}{=} \mathbb{C}^3 \times \ell^1_{\nu_2} \times L^1_{\nu},
\]

with the product norm

\[
\| x \| = \|(\lambda_1, \lambda_2, a, y, z)\| \overset{\text{def}}{=} \max\{ |\lambda_1|, |\lambda_2|, |a|, \|y\|_{\ell^1_{\nu_2}}, \|z\|_{L^1_{\nu}} \}. \tag{4.19}
\]

At times, we will also use the so-called component norm

\[
\| x \|_c \overset{\text{def}}{=} (\max\{|\lambda_1|, |\lambda_2|, |a|, \|y\|_{\ell^1_{\nu_2}}, \|z\|_{L^1_{\nu}} \}) \in \mathbb{R}^3. \tag{4.20}
\]

**Definition 4.2.4.** The conjugate \( x^* \) of \( x \in X \) is defined by

\[
x^* \overset{\text{def}}{=} (\bar{\lambda}_1, \bar{\lambda}_2, \bar{a}, \bar{y}^*, \bar{z}^*),
\]

with the conjugate of \( y \) and \( z \) defined by

\[
y_k^* \overset{\text{def}}{=} \bar{y}_{-k}, \quad k \in \mathbb{Z},
\]

\[
z_{jk}^* \overset{\text{def}}{=} \bar{z}_{-j-k}, \quad j, k \in \mathbb{Z}.
\]

Let \( Y \) be a space on which conjugate symmetry is defined, such as \( X, \ell^1_{\nu_2}, \) or \( L^1_{\nu} \). Then the subspace of conjugate symmetric elements is

\[
S(Y) \overset{\text{def}}{=} \{ x \in Y : x = x^* \}.
\]
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**Definition 4.2.5.** A map $T : X \to X$ is said to preserve conjugate symmetry if $T(S(X)) \subseteq S(X)$.

For $\nu' < \nu$, we have

$$
\left( \begin{array}{c}
G^x \\
F
\end{array} \right) : \mathbb{C}^3 \times \ell^1_{\nu_2} \times \mathcal{L}^1_{\nu} \to \mathbb{C}^4 \times \ell^1_{\nu_2} \times \mathcal{L}^1_{\nu'}
$$

$$
(\lambda_1, \lambda_2, a, y, z) \mapsto (v', y', z').
$$

Furthermore $(F_1)_k$ and $(F_2)_{jk}$ are trivially zero for $k = 0$ and $(j, k) = (0, 0)$, that is $y'_0 = 0$ and $z'_{00} = 0$. Therefore we can concatenate a projection

$$
\tilde{\Pi} : \mathbb{C}^n \times \ell^1_{\nu_2} \times \mathcal{L}^1_{\nu} \to \mathbb{C}^{n-2} \times \ell^1_{\nu_2} \times \mathcal{L}^1_{\nu'}
$$

$$
(v', y', z') \mapsto (v'', y'', z'')
$$

with $v''_i = v'_i$ for $i = 1, \ldots, n - 2$ and

$$
y''_k = \begin{cases} v'_{n-1} & \text{if } k = 0, \\ y'_k & \text{if } k \neq 0, \end{cases} \quad \text{and} \quad z''_{jk} = \begin{cases} v'_n & \text{if } (j, k) = (0, 0), \\ z'_{jk} & \text{if } (j, k) \neq (0, 0). \end{cases}
$$

In this way we remove the trivial identities from the formulation and arrive at

$$
H^x(x) \overset{\text{def}}{=} \tilde{\Pi} \left( \begin{array}{c}
G^x \\
F
\end{array} \right) (x) = 0,
$$

which generically has a one-dimensional solution set.

Let $x = (\lambda_1, \lambda_2, a, y, z) \in X$ and $w = (\beta_1, \beta_2, b, r, s) \in X$, then we extend the notation $\langle \cdot, \cdot \rangle$ to $X$ by defining

$$
\langle x, w \rangle \overset{\text{def}}{=} \lambda_1 \beta_1 + \lambda_2 \beta_2 + ab + \langle y, r \rangle + \langle z, s \rangle.
$$

In the following, we will assume to have computed two numerical solutions $\hat{x}_0 = (\hat{\lambda}_{10}, \hat{\lambda}_{20}, \hat{a}_0, \hat{y}_0, \hat{z}_0) \in S(X)$ and $\hat{x}_1 = (\hat{\lambda}_{11}, \hat{\lambda}_{21}, \hat{a}_1, \hat{y}_1, \hat{z}_1) \in S(X)$ of (4.18). Assuming the two numerical solutions are close to each other, i.e. $\|\hat{x}_0 - \hat{x}_1\| < 1$, it is reasonable to expect that the segment connecting them is an approximation of a nearby solution branch. To construct a one-to-one correspondence between the point on the segment $[\hat{x}_0, \hat{x}_1]$ and the solution branch $x(s), s \in [0, 1]$, we follow [20] and add to (4.18) the so-called continuation equation

$$
E_s(x) = \langle x, q_s \rangle - c_s,
$$

where

$$
q_s = q_0 + s(q_1 - q_0), \quad \text{and} \quad c_s = c_0 + s(c_1 - c_0),
$$

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with
\[ g_s = \frac{(v_s + v_s^*)}{2}, \quad \text{and} \quad c_s = \langle \hat{x}_s, q_s \rangle, \quad \text{for } s = 0, 1, \quad (4.23) \]
where \( v_s \) is approximately in \( \ker(DH^{\hat{x}_s}(\hat{x}_s)) \), with \( H^{\hat{x}_s}(x) \) defined in (4.21).
Furthermore, we assume \( q_s \), for \( s = 0, 1 \), has finitely many non-zero Fourier coefficients, as do \( \hat{x}_0 \) and \( \hat{x}_1 \). The continuation problem is then
\[ H_s(\lambda_1, \lambda_2, a, y, z) = \tilde{I} \begin{pmatrix} E_s(\lambda_1, \lambda_2, a, y, z) \\ G^{\hat{x}_s}(\lambda_1, \lambda_2, a, y, z) \\ F(\lambda_1, \lambda_2, a, y, z) \end{pmatrix}. \quad (4.24) \]

Remark 4.2.6. It follows from the definition of \( H_s \) in (4.24), that
\[ H_s(x^*) = H_s(x)^*, \]
where the codomain of \( H_s \) is \( X' = \mathbb{C}^3 \times \ell^1_{\nu_2} \times \mathcal{L}^1_{\nu'} \) for \( \nu' < \nu \). Therefore, \( H_s \) preserves conjugate symmetry.

We are interested in solving the continuation problem
\[ H_s : X \to X', \quad H_s(x) = 0 \quad \forall s \in [0, 1]. \quad (4.25) \]

Remark 4.2.7. A solution branch \( x(s) \) of (4.25) goes through a Hopf bifurcation if there exists \( s^*, s_0, s_1 \) such that \( s^* \in (s_0, s_1), \ a(s^*) = 0 \) and \( a(s) \neq 0 \) for \( s \in [s_0, s_1]\setminus\{s^*\} \). The second statement guarantees that the time-dependent term \( a z \) does not vanish along the rest of the branch.

By continuity of \( a(s) \), if it holds that \( a(s_0) \cdot a(s_1) < 0 \) then there exists a \( s^* \in (s_0, s_1) \) such that \( a(s^*) = 0 \). We are thus left with proving that \( a(s) \) changes sign along the validated solution branch and \( a'(s^*) \neq 0 \), in order to prove the existence of a Hopf bifurcation of the Kuramoto-Sivshinsky equation.

Remark 4.2.8. We can choose the continuation equation (4.22) conveniently to simplify the verification in Remark 4.2.7. Indeed, the choice
\[ q_s = (0, 0, 1, 0, 0) \in X \quad \text{and} \quad c_s = \hat{a}_s \]
in essence turns the general (pseudo-arclength) continuation into parameter continuation with respect to \( a \). In this case, if the validation succeeds we immediately conclude that \( a'(s) \neq 0 \), provided \( \hat{a}_0 \neq \hat{a}_1 \). As a consequence, if we can prove that \( a(s) \) changes sign along the validated curve of solutions, Remark 4.2.7 implies the existence of a Hopf bifurcation.
4.3 The radii polynomial approach

For the parameter dependent zero finding problem (4.25), we aim to validate the segment between the two numerical solutions \( \hat{x}_0 \) and \( \hat{x}_1 \). The numerical solutions are such that

\[
H_s(\hat{x}_s) \approx 0, \quad \hat{x}_s \in S(X) \quad \text{for} \quad s = 0, 1.
\]

For \( s \in [0, 1] \), we introduce the notation

\[
\hat{x}_s = \hat{x}_0 + s(\hat{x}_1 - \hat{x}_0).
\]

Following [20], we transform the zero finding problem (4.25) into a parameter dependent fixed point problem

\[
T_s(x) = x,
\]

with \( s \in [0, 1] \) and \( T_s \) the Newton-Kantorovich operator defined by

\[
T_s : X \to X
\]

\[
x \mapsto x - A_s H_s(x),
\]

where \( A_s \) is an approximation of \( D H_s(\hat{x}_s)^{-1} \) for all \( s \) in \( [0, 1] \). The linear operator \( A_s \) will be constructed in Section 4.6 satisfying the following conditions:

\[
A_s H_s(x) \in X \quad \text{for} \quad x \in X
\]

\[
A_s \text{ is injective for all } s \in [0, 1].
\]

In order to prove that \( T_s \) is a contraction in a neighborhood of \( \hat{x}_s \), we first introduce

\[
B_r(x) \overset{\text{def}}{=} \{ y \in X : \| y - x \|_X \leq r \}.
\]

To prove that \( T_s \) is a contraction in \( B_r(\hat{x}_s) \) for some \( r \), we have to determine the bounds

\[
Y_s \geq \max_{s \in [0, 1]} \| T_s(\hat{x}_s) - \hat{x}_s \|_X,
\]

and

\[
Z_s(r) \geq \max_{s \in [0, 1]} \sup_{b, c \in B_1(0) \subset X} \| D T_s(\hat{x}_s + rb) c \|_X.
\]

With \( Y_s \) and \( Z_s(r) \) computed rigorously, we can then apply the following theorem and validate our numerical approximations.
Theorem 4.3.1. Let $A_s$ satisfy the condition (4.28), let $T_s : X \to X$ be defined as in (4.26), let $Y_s$ and $Z_s(r)$ be defined as in (4.29) and (4.30) respectively. Let the radii polynomial be defined as

$$p(r) \overset{\text{def}}{=} Y_s + Z_s(r) - r.$$  

If there exists $0 < r^* \in \mathbb{R}$ such that $p(r^*) < 0$ then there exists a continuous curve $\{x(s)\}_{s \in [0,1]} \subset X$ such that $x(s) \in B_{r^*}(\hat{x}_s)$ and $H_s(x(s)) = 0$ for all $s \in [0,1]$. Furthermore, if $\hat{x}_s, q_s \in \mathcal{S}(X)$, then $x(s) \in \mathcal{S}(X)$.

Proof. The first part of this Theorem has been presented and proven in many contexts, such as in [20] and [74].

The symmetry argument follows from the following considerations. Thanks to the injectivity of $A_s$, $T_s(x(s)) = x(s)$ implies $H_s(x(s)) = 0$. We have $H_s(x(s))^* = H_s(x(s))^* = 0$ and thus $T_s(x(s))^* = x(s)^*$.

Since $\hat{x}_s \in \mathcal{S}(X)$, the ball $B_{r^*}(\hat{x})$ is invariant under conjugate symmetry, hence $x(s)^* \in B_{r^*}(\hat{x})$. Due to uniqueness of the fixed point of $T_s$ in $B_{r^*}(\hat{x})$, we conclude that $x(s)^* = x(s) \in \mathcal{S}(X)$. 

If the continuation equation is chosen as in Remark 4.2.8, the solution curve $\{x(s)\}_{s \in [0,1]} \subset X$ is smooth.

4.4 Linear operators

4.4.1 Operators on $X$

For simplicity, we introduce the notation $X = X_1 \times X_2 \times X_3$, where $X_1 = \mathbb{C}^3$, $X_2 = \ell^1_{\nu_2}$, $X_3 = \mathcal{L}_1$. Let $B \in B(X, X)$,

$$B : X \to X$$

$$x = (a, b, c) \mapsto Bx = (a', b', c'),$$

which we can write as

$$\begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

with $B_{ij} \in B(X_i, X_j)$.

We denote by $\| \cdot \|_{ij}$ the operator norm induced by the norms on $X_i$ and $X_j$. In particular, all the operator norms here introduced have a simple explicit
form or bound:

\[
\|B_{11}\|_{11} = \max_{n=1,2,3} \sum_{m=1,2,3} |B_{nm}|, \\
\|B_{12}\|_{12} = \max_{n=1,2,3} \sup_{k \in \mathbb{Z}} \nu^{-|k|} |B_{nk}|, \\
\|B_{13}\|_{13} = \max_{n=1,2,3} \sup_{(j,k) \in \mathbb{Z}^2} \nu^{-|k|}-|j| |B_{nk}|, \\
\|B_{21}\|_{21} \leq \sum_{m=1,2,3} \sum_{k \in \mathbb{Z}} \nu^{|k|} |B_{km}|, \\
\|B_{22}\|_{22} = \sup_{k_1 \in \mathbb{Z}} \nu_{2}^{-|k_1|} \sum_{k \in \mathbb{Z}} \nu_{1}^{|k|} |B_{k_k}|, \\
\|B_{31}\|_{31} \leq \sum_{m=1,2,3} \sum_{k \in \mathbb{Z}^2} \nu^{|k|}+|j| |B_{km}|, \\
\|B_{32}\|_{32} = \sup_{k_1 \in \mathbb{Z}} \nu_{2}^{-|k_1|} \sum_{j,k \in \mathbb{Z}} \nu_{1}^{|j|} \nu_{2}^{|k|} |B_{jkk}|, \\
\|B_{33}\|_{33} = \sup_{j_1,k_1 \in \mathbb{Z}} \nu_{2}^{-|j_1|} \nu_{2}^{-|k_1|} \sum_{j,k \in \mathbb{Z}} \nu_{1}^{|j|} \nu_{2}^{|k|} |B_{jjkk}|.
\]

(4.32)

\(\|B\|_{23}\) is not presented because it is never used in this paper.

On \(B(X,X')\), we introduce the component norm

\[
\|B\|_c = \left( \begin{array}{c}
\|B_{11}\|_{11} \\
\|B_{12}\|_{12} \\
\|B_{13}\|_{13} \\
\|B_{21}\|_{21} \\
\|B_{22}\|_{22} \\
\|B_{23}\|_{23} \\
\|B_{31}\|_{31} \\
\|B_{32}\|_{32} \\
\|B_{33}\|_{33}
\end{array} \right) \in \mathbb{R}^{3 \times 3}.
\]

By the definition of operator norm on the space \(X\), it holds

\[
\|B\| = \max_{j=1,\ldots,3} \sum_{i=1,\ldots,3} \|B_{ij}\|_{ij}.
\]

(4.33)

4.4.2 Convolution operators on \(\ell_{\nu_2}^1\) and \(L_{\nu}^1\)

Let us consider \(u\), an element of \(\ell_{\nu_2}^1\). We introduce the convolution operator as the operator \(C_{\ell_{\nu_2}^1} : \ell_{\nu_2}^1 \to B(\ell_{\nu_2}^1, \ell_{\nu_2}^1)\) that satisfies

\[
C_{\ell_{\nu_2}^1}(u)w = u \ast w \quad \text{for all } u, w \in \ell_{\nu_2}^1.
\]

(4.34)

By the definition of convolution, it follows that, for \(u = (u_k)_{k \in \mathbb{Z}} \in \ell_{\nu_2}^1\), the matrix representation of \(C_{\ell_{\nu_2}^1}\) is

\[
C_{\ell_{\nu_2}^1}(u)_{k_1k_2} = u_{k_1-k_2}.
\]
It follows from the Banach algebra property that
\[
\|C_{\ell^1_{\nu_2}}(u)\|_{B(\ell^1_{\nu_2},\ell^1_{\nu_2})} = \|u\|_{\ell^1_{\nu_2}}.
\] (4.35)

Following the same structure in $\mathcal{L}^1_{\nu}$, we introduce the convolution operator $C_{\mathcal{L}^1_{\nu}} : \mathcal{L}^1_{\nu} \to B(\mathcal{L}^1_{\nu},\mathcal{L}^1_{\nu})$ such that
\[
C_{\mathcal{L}^1_{\nu}}(u)w = u * w \quad \text{for all } u, w \in \mathcal{L}^1_{\nu}.
\] (4.36)

By the definition of convolution, the matrix representation of $C_{\mathcal{L}^1_{\nu}}(u)$ is
\[
C_{\mathcal{L}^1_{\nu}}(u)_{j_1,j_2,k_1,k_2} = u_{j_1 - j_2,k_1 - k_2},
\]
and it follows from the Banach algebra property that
\[
\|C_{\mathcal{L}^1_{\nu}}(u)\|_{B(\mathcal{L}^1_{\nu},\mathcal{L}^1_{\nu})} = \|u\|_{\mathcal{L}^1_{\nu}}.
\] (4.37)

### 4.4.3 Finite dimensional projection

In order to study a finite dimensional subset of $X$, we define for any $K = (M,N) \in \mathbb{N}^2$ the finite set of indices smaller than $K$ as
\[
\mathcal{F}_K = \mathcal{F}_{(M,N)} = \{(j,k) \in \mathbb{Z}^2 \mid |j| \leq M, |k| \leq N\}.
\] (4.38)

The projection on the first $K = (M,N) \in \mathbb{N}^2$ modes is defined as
\[
\Pi^{(K)} : X = \mathbb{C}^3 \times \ell^1_{\nu_2} \times \mathcal{L}^1_{\nu} \to X
\]
\[
x = (a,b,c) \mapsto \Pi^{(K)} x = (a,\Pi^{(K)}_1 b,\Pi^{(K)}_2 c)
\] (4.39)

with
\[
(\Pi^{(K)}_1 b)_k = \begin{cases} b_k & \text{if } |k| \leq N, \\ 0 & \text{if } |k| > N, \end{cases}
\] (4.40)

and
\[
(\Pi^{(K)}_2 c)_{jk} = \begin{cases} c_{jk} & \text{if } (j,k) \in \mathcal{F}_{(M,N)}, \\ 0 & \text{if } (j,k) \notin \mathcal{F}_{(M,N)}. \end{cases}
\] (4.41)

If $x = \Pi^{(K)} x \in X$, we say that $x$ has at most $K$ non-zero modes, and similarly for $b = \Pi^{(K)}_1 b \in \ell^1_{\nu_2}$ and $c = \Pi^{(K)}_2 c \in \mathcal{L}^1_{\nu}$. The complementary projection on the infinite tail is given by
\[
\Pi^{(K)}_\infty x = x - \Pi^{(K)} x,
\] (4.42)
where the notation is meant to indicate the projection from the \((K + 1)\)-th mode onwards. There is a natural correspondence between \(\Pi_1^{(K)}\ell^1_{\nu_2}\) and \(\mathbb{C}^{2N+1}\), which we denote by the truncation operator

\[
\tau_1^{(K)} : \ell^1_{\nu_2} \to \mathbb{C}^{2N+1}, \\
b \mapsto (b_k)_{k=-N,...,N},
\]

(4.43)
as well as between \(\Pi_2^{(K)}\mathcal{L}^1_{\nu}\) and \(\mathbb{C}^{(2M+1)\times (2N+1)}\), which we indicate by

\[
\tau_2^{(K)} : \mathcal{L}^1_{\nu} \to \mathbb{C}^{(2M+1)\times (2N+1)}, \\
c \mapsto (c_{j,k})_{(j,k) \in \mathbb{F}_K}.
\]

(4.44)

We introduce the notation for truncation in \(X\) as

\[
\tau^{(K)} : X \to \mathbb{C}^3 \times \mathbb{C}^{2N+1} \times \mathbb{C}^{(2M+1)\times (2N+1)}
\]

(4.47)

\[
x = (a, b, c) \mapsto \tau^{(K)}x = (a, \tau_1^{(K)}b, \tau_2^{(K)}c).
\]

(4.48)

Its corresponding inverse is denoted by

\[
E : \mathbb{C}^3 \times \mathbb{C}^{2N+1} \times \mathbb{C}^{(2M+1)\times (2N+1)} \to \Pi^{(K)}X.
\]

Definition 4.4.1. A \(A \in \mathcal{B}(X, X)\) is a finite linear operator of center dimensions \(K_1\) and \(K_2\) if

\[
Ax = \Pi^{(K_1)}(A\Pi^{(K_2)}x), \quad \text{for all } x \in X.
\]

If \(K_1 = K_2 = K\), we say that \(A\) has center dimension \(K\).

Remark 4.4.2. A finite linear operator \(A \in \mathcal{B}(X, X)\) of center dimensions \(K_1 = (M_1, N_1)\) and \(K_2 = (M_2, N_2)\) can be represented by a complex matrix of dimension \((3 + (2N_2 + 1) + (2M_2 + 1)(2N_2 + 1)) \times (3 + (2N_1 + 1) + (2M_1 + 1)(2N_1 + 1))\).

A class of operators we will be working with extensively is the class of eventually diagonal operators in \(X\). It is an extention of the definition provided in [20] for eventually diagonal operators in \(\ell^1_{\nu}\).

Definition 4.4.3. Let \(A : X \to X\) be a linear operator. \(A\) is eventually diagonal of center dimension \(K\) if it can be decomposed into \(A^F\) and \(A^I\) such that

\[
Ax = A^F x + A^I \Pi^{(K)}_{(K)} y
\]

with \(A^I\) a diagonal operator and \(A^F\) a finite linear operator of center dimension \(K\). The same definition applies for \(A : \ell^1_{\nu} \to \ell^1_{\nu}\) and \(A : \mathcal{L}^1_{\nu} \to \mathcal{L}^1_{\nu}\).
Lemma 4.4.4. If $A$ is eventually diagonal,
\[
\|A_{ij}\|_{ij} \leq \max\{\|A^F_{ij}\|_{ij}, \|A^I_{ij}\|_{ij}\},
\]
for $i, j = 1, 2, 3$.

Proof. Let $A = A^I + A^F$ be an eventually diagonal operator of center dimension $K = (M, N)$. For $i \neq j$, $A^I_{ij} = 0$, and $A^I_{i1} = 0$, hence we are left to compute $\|A_{22}\|_{22}$ and $\|A_{33}\|_{33}$. By the definition of the norms in (4.32), we have
\[
\|A_{22}\|_{22} = \sup_{k_2 \in \mathbb{Z}} \nu_2^{-|k_2|} \sum_{k_1 \in \mathbb{Z}} \nu_2^{|k_1|} |(A_{22})_{k_1 k_2}| \\
= \max\{ \sup_{|k_2| \leq N} \nu_2^{-|k_2|} \sum_{k_1 \in \mathbb{Z}} \nu_2^{|k_1|} |(A_{22})_{k_1 k_2}|, \sup_{|k_2| > N} \nu_2^{-|k_2|} \sum_{k_1 \in \mathbb{Z}} \nu_2^{|k_1|} |(A_{22})_{k_1 k_2}| \} \\
= \max\{ \sup_{|k_2| \leq N} \nu_2^{-|k_2|} \sum_{k_1 \in \mathbb{Z}} \nu_2^{|k_1|} |(A^F_{22})_{k_1 k_2}|, \sup_{|k_2| > N} |(A^I_{22})_{k_1 k_1}| \} \\
= \max\{\|A^F_{22}\|_{22}, \|A^I_{22}\|_{22}\}.
\]
The same approach can be applied to $A_{33}$, thus concluding the proof. \qed

Remark 4.4.5. The product of two eventually diagonal operators of center dimensions $K_1$ and $K_2$, respectively, is an eventually diagonal operator of center dimension
\[
K = \max\{K_1, K_2\}.
\]
The operator $A$ introduced in Section 4.3 will be constructed to be an eventually diagonal operator. We therefore want to study this type of operators a little more in depth.

Remark 4.4.6. Let $x = \Pi^{(K')} x$, $x \in X$, and let $A$ be an eventually diagonal operator of center dimension $K$, then
\[
Ax = \Pi^{(\bar{K})}(Ax),
\]
with
\[
\bar{K} = \max(K, K').
\]

Another case in which it is easy to compute the number of non-zero modes is when we are considering convolutions.
Lemma 4.4.7. Let \( x = \Pi_2^{(K_1)} x \) and \( v = \Pi_2^{(K_2)} v \) then
\[
x \ast v = \Pi_2^{(K_1 + K_2)} (x \ast v).
\]
The same holds in \( \ell_{\nu_2}^1 \).

We also introduce the definition of a \( K \)-diagonal operator.

**Definition 4.4.8.** \( B : X \rightarrow X \) is called a \( K \)-diagonal operator on \( X \) if
\[
By = \Pi^{(K + K')} (By)
\]
for all \( y \in X \) such that \( y = \Pi^{(K')} y \).

Similarly, if \( B : \mathcal{L}_\nu^1 \rightarrow \mathcal{L}_\nu^1 \) and \( By = \Pi_2^{(K + K')} (By) \) for any \( y \in \mathcal{L}_\nu^1 \) with \( K' \) non-zero modes, then \( B \) is a \( K \)-diagonal operator in \( \mathcal{L}_\nu^1 \), and the same definition can be applied to operators on \( \ell_{\nu_2}^1 \).

**Remark 4.4.9.** It is obvious by the definition that diagonal operators are 1-diagonal operators. Furthermore, by Lemma 4.4.7, the convolution operators associated to \( y \in \mathcal{L}_\nu^1 \) with \( K \) non-zero modes are \( K \)-diagonal.

**Lemma 4.4.10.** \( B \) is a \( K \)-diagonal operator in \( \mathcal{L}_\nu^1 \) if and only if its matrix representation \( B \) is such that \( B_{jj_1kk_1} = 0 \) if \( j - j_1, k - k_1 \) \( \notin F_K \). A similar statement holds in \( \ell_{\nu_2}^1 \).

**Lemma 4.4.11.** Let \( K = (M, N) \in \mathbb{N}^2 \), and let \( B : X \rightarrow X \) be a \( \tilde{K} \)-diagonal linear operator. Then \( B_{22} : \ell_{\nu_2}^1 \rightarrow \ell_{\nu_2}^1 \) is a \( K \)-diagonal linear operator and so is \( B_{33} : \mathcal{L}_\nu^1 \rightarrow \mathcal{L}_\nu^1 \). For \( B_{32} : \ell_{\nu_2}^1 \rightarrow \mathcal{L}_\nu^1 \), its matrix representation \( B_{32} \) is such that \( (B_{32})_{jkk_1} = 0 \) if \( |k - k_1| > N \).

We do not discuss \( B_{23} \) because vanishes for all the operators treated in this paper.

**Theorem 4.4.12.** Let \( A : X \rightarrow X \) be a finite dimensional operator of center dimension \( K_1 = (M_1, N_1) \) and \( K_2 = (M_2, N_2) \). Let \( B : X \rightarrow X \) be a \( \tilde{K} \)-diagonal operator. Then the operator product \( AB \) is a finite dimensional operator of center dimension \( K_1 + K_2 \).

**Proof.** The proof will be carried out for \( A' = A_{22} \) and \( B' = B_{22} \), but its extension to the whole of \( X \) follows the same reasoning.

By definition of the operator product, it holds that
\[
(A' B')_{k_1k_2} = \sum_{k_3 \in \mathbb{Z}} A'_{k_1k_3} B'_{k_3k_2}.
\]
The assumptions imply that \( A'_{k_1k_3} = 0 \) if \( k_1 > N_1 \) or \( k_3 > N_1 \). Furthermore, \( B'_{k_3k_2} = 0 \) if \( |k_3 - k_2| > \tilde{N} \). Hence, we can rewrite

\[
(A'B')_{k_1k_2} = \sum_{k_3 \in I} A'_{k_1k_3} B'_{k_3k_2},
\]

with

\[
I = \{k_2 - \tilde{N}, \ldots, k_2 + \tilde{N}\} \cap \{-N_2, \ldots, N_2\}.
\]

The set \( I \) is non-empty only if \( |k_2| \leq N_2 + \tilde{N} \). This concludes the proof.

**Corollary 4.4.13.** Under the hypothesis of Theorem 4.4.12, we have

\[
AB = \Pi^{(K_1)} A \Pi^{(K_2)} B = \Pi^{(K_1)} A \Pi^{(K_2)} B \Pi^{(K_2 + \tilde{K})}.
\]

Hence, the operator norm \( \|AB\| \) is explicitly computable, since it depends on just a finite number of indices of the matrix representation of \( B \), namely on \( \Pi^{(K_2)} B \Pi^{(K_2 + \tilde{K})} \).

### 4.5 Construction of the operator \( A_s \)

In this Section, we construct the operator \( A_s \) needed for the application of the radii polynomial approach to the Kuramoto-Sivashinsky PDE given by (4.24). We consider the segment

\[
\hat{x}_s = E\hat{x}_0 + s(E\hat{x}_1 - E\hat{x}_0),
\]

where \( \hat{x}_0, \hat{x}_1 \in \tau^{(K)} S(X) \) are the two numerical approximate solutions of (4.24) already introduced at the end of Section 4.2.

The radii polynomial approach to the validation of a branch relies strongly on finding a reasonably good approximation \( A_s \) of the inverse of \( DH_s(\hat{x}_s) \). To construct \( A_s \), we will introduce the linear operator \( A'_s \), an approximation of \( DH_s(\hat{x}_s) \). First, let us compute explicitly

\[
DH_s(x) = \Pi \begin{pmatrix}
D_{(\lambda_1, \lambda_2, a)} \left( E_s(x) \right) & D_y \left( E_s(x) \right) & D_z \left( E_s(x) \right) \\
D_{(\lambda_1, \lambda_2, a)} F_1(x) & \lambda_2 K^4 - K^2 - 2iKC^{v_2}(y) & 0 \\
D_{(\lambda_1, \lambda_2, a)} F_2(x) & -2\lambda_1 iKC^{v_2} (z) E_\ell & iJ + \lambda_1 \lambda_2 K^4 - \lambda_1 K^2 - 2\lambda_1 iKC^{v_2}(E_{\ell y + \ell z})
\end{pmatrix}.
\]
with

\[ D(\lambda_1, \lambda_2, a)F_1(x) = \begin{pmatrix} 0 & K^4y & 0 \\ \lambda_2K^4-K^2z & 0 & -\lambda_1z \end{pmatrix}, \]

\[ D(\lambda_1, \lambda_2, a)F_2(x) = \begin{pmatrix} (\lambda_2K^4-K^2z) & \lambda_1K^4z & -\lambda_1z \end{pmatrix}, \]

We define

\[ \hat{H}_s(x) \overset{\text{def}}{=} \tau^{(K)}H_s(\tau^{(K)}x), \]

which can be interpreted as a map from \( \mathbb{C}^{3+(2N+1)+(2N+1)(2M+1)} \) to itself, and we have

\[ D\hat{H}_s(x) = \tau^{(K)}D\hat{H}_s(\tau^{(K)}x)\tau^{(K)}, \] (4.50)

which can be represented by a complex square matrix of dimensions \( 3 + (2N+1) + (2N+1)(2M+1) \).

We construct the dependence of \( A_s^I \) on \( s \) to be linear:

\[ A_s^I = A_0^I + s(A_1^I - A_0^I), \]

where we define \( A_0^I \) as

\[ A_0^I = ED\hat{H}_s(\tilde{x}_s) + A_1^{IJ}, \quad \text{for } s = 0, 1, \]

with \( A_1^{IJ} \) the tail operator

\[ A_1^{IJ} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \hat{\lambda}_2K^4 - K^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Pi_{(K)}^{\infty}. \]

For the approximate inverse \( A_s \) of \( A_s^I \), we set

\[ A_s = A_0 + s(A_1 - A_0), \]

and

\[ A_s = A_s^F + A_s^I, \quad \text{for } s = 0, 1, \]

with

\[ A_s^F = EQ_s\tau^{(K)}, \quad \text{for } s = 0, 1, \]

where \( Q_s \) is an approximate numerical inverse of the matrix representation of \( D\hat{H}_s(\tau^{(K)}x_s) \), a complex square matrix of dimension \( 3 + (2N+1) + (2N+1)(2M+1) \). The diagonal operators are defined as

\[ A_s^I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A_{s1}^I & 0 \\ 0 & 0 & A_{s2}^I \end{pmatrix} \Pi_{(K)}^{\infty}, \] (4.51)
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with

\[ A^I_{s1} = (\tilde{\lambda}_{2s} K^4 - K^2)^{-1} \]

and

\[ A^I_{s2} = (iJ + \tilde{\lambda}_{1s} \tilde{\lambda}_{2s} K^4 - \tilde{\lambda}_{1s} K^2)^{-1} \]

where the inverse here is interpreted elementwise, that is, for \((j, k) \notin \mathbf{F}_K\),

\[ (A^I_{s1}y)_k = \frac{yk}{\lambda_{2s} k^4 - k^2}, \]

and

\[ (A^I_{s2}z)_{jk} = \frac{z_{jk}}{ij + \tilde{\lambda}_{1s} \tilde{\lambda}_{2s} k^4 - \tilde{\lambda}_{1s} k^2}. \]

In Equation (4.28), we stated two conditions which \(A_s\) needs to satisfy. First, we require that \(A_s H_s(x) \in X\) for all \(x \in X\). Although \(H_s(x)\) is not in \(X\) due to the unboundedness of the operators \(J\) and \(K\), this is counteracted by the action of the tail of \(A_s\), hence \(A_s H_s(x) \in X\).

The second condition is that \(A_s\) is injective. This condition is proven \textit{a posteriori} by the radii polynomial approach. Indeed, if the validation succeeds, it holds that \(1 > Z(r) \geq \sup_{s \in [0, 1]} \|I - A_s A^I_s\|\), thus \(A_s\) is surjective for all \(s \in [0, 1]\). We also know that \(A_s\) is eventually diagonal for all \(s\), that is \(A_s = A^F_s + A^I_s\). By surjectivity of \(A_s\), we deduce surjectivity of \(A^F_s\), and since \(A^F_s\) is finite dimensional, surjectivity implies injectivity. Finally injectivity of \(A^I_s\) follows from the nontriviality of the elements on its diagonal.

Remark 4.5.1. The elements on the diagonal are nontrivial if \(\tilde{\lambda}_{2s} k^4 - k^2 \neq 0\) for all \(|k| > N\). Thus we impose

\[ N^2 > \max\{\tilde{\lambda}_{20}^{-1}, \tilde{\lambda}_{21}^{-1}\}. \]

4.6 Bounds for the norm of eventually diagonal operators

This section is devoted to bound the norm of some eventually diagonal operators, where we add some constraint on the diagonal part of the operator. Let \(P\) be an eventually diagonal operator in \(B(\mathcal{L}_1^1, \mathcal{L}_1^1)\) of center dimension \(K = (M, N) \in \mathbb{N}^2\), then it has the form

\[ P : \mathcal{L}_1^1 \rightarrow \mathcal{L}_1^1, \]

\[ u \mapsto P : (Pu)_{jk} = \begin{cases} E(B\tau_{jk}^{\mathcal{F}} u)_{jk} & \text{for } (j, k) \in \mathbf{F}_K, \\ p(j, k)u_{jk} & \text{otherwise}, \end{cases} \]
for a function $p$ and a matrix $B \in \mathbb{C}^{(2M+1)(2N+1)\times(2M+1)(2N+1)}$, applying Remark 4.4.2. In this section we will study the norm of $P$ when $p$ is a rational polynomial of the form

$$p(j, k) = \frac{q(j, k)}{p(j, k)} = \frac{q_1(j) + q_2(k)}{1p_1(j) + p_2(k)},$$

with $p_1(j)$ and $p_2(k)$ real-valued polynomials and $q_1(j)$ and $q_2(k)$ complex polynomials. If there is any constant term in $q$, it will be assigned to $q_2$.

It holds, by Lemma (4.4.4) and by the definition of the operator norm on $L^p_1$, (4.32), that

$$\|P\| = \max\{\|B\|, \sup_{(j, k) \notin \mathbb{F}_K} |p(j, k)|\}.$$

**Remark 4.6.1.** The bound on $\sup_{(j, k) \notin \mathbb{F}_K} |p(j, k)|$ can be finite only if the order of $q_i$ is smaller or equal to the order of $p_i$, for $i = 1, 2$.

To simplify the presentation, we will further restrict ourselves to

$$p_1(j) = j.$$

Most results presented can be easily generalised to any real polynomial $p_1(j)$, but it will not be done here.

By the triangle inequality

$$\sup_{(j, k) \notin \mathbb{F}_K} |p(j, k)| \leq \sup_{(j, k) \notin \mathbb{F}_K} \left| \frac{q_1(j)}{1j + p_2(k)} \right| + \sup_{(j, k) \notin \mathbb{F}_K} \left| \frac{q_2(k)}{1j + p_2(k)} \right|$$

(4.52)

We will consider the two fractions separately. With the restrictions imposed on $q_1$, we have

$$q_1(j) = c_1j.$$

Therefore the first fraction is bounded trivially by $|c_1|$.

**Remark 4.6.2.** If there exists a $(j^*, k^*) \notin \mathbb{F}_K$ such that $p(j^*, k^*) = 0$, then the norm of $P$ is infinite. Obviously, $j^* = 0$, hence in the following we want to ensure that there is no $k^* \in \mathbb{N}$, such that $|k^*| > N$ and $p(0, k^*) = 0$. This is incorporated in the definition of $Q_N$ below in (4.57), which will simply be infinity if this is violated.

For the second bound, we remark that, for fixed $k$, $\left| \frac{q_2(k)}{1j + p_2(k)} \right|$ decreases if $|j|$ increases, hence

$$\sup_{(j, k) \notin \mathbb{F}_K} \left| \frac{q_2(k)}{1j + p_2(k)} \right| \leq \max \left\{ \max_{|j| = M+1, |k| \leq N+1} \left| \frac{q_2(k)}{1j + p_2(k)} \right|, \sup_{|k| > N} \left| \frac{q_2(k)}{p_2(k)} \right| \right\},$$

(4.53)
The first maximum of (4.53) is taken over a finite set. Therefore it is directly computable. The supremum, on the other hand, is computed over an infinite set and requires a more refined approach.

We want to bound

\[ \sup_{|k| > N} \left| \frac{q_2(k)}{p_2(k)} \right| \]  

with a finite number of computations. As a first step, we construct \( k_1' \in \mathbb{N} \) as the integer upper bound of all vertical asymptotes, satisfying

\[ p_2(x) \neq 0 \quad \text{for all } |x| > k_1', \tag{4.55} \]

and an integer upper bound \( k_2' \) on all local maxima and minima, satisfying

\[ q_2'(x)p_2(x) - p_2'(x)q_2(x) = 0 \quad \text{for all } |x| > k_2'. \tag{4.56} \]

We use the minimum of Lagrange’s and Cauchy’s bounds, which provide explicit finite bounds on roots of polynomials. Then we set \( k' = \max\{k_1', k_2'\} \).

It then holds that (4.54) is bounded by

\[ Q_N \overset{\text{def}}{=} \max \left\{ \lim_{k \to \pm \infty} \left| \frac{q_2(k)}{p_2(k)} \right|, \max_{|k| > N, |k| \leq k'} \left| \frac{q_2(k)}{p_2(k)} \right|, \left| \frac{q_2(N + 1)}{p_2(N + 1)} \right|, \left| \frac{q_2(-N - 1)}{p_2(-N - 1)} \right| \right\}. \tag{4.57} \]

The limit \( \lim_{k \to \pm \infty} \left| \frac{q_2(k)}{p_2(k)} \right| \) is trivial to be determined and is non-zero only if \( q_2 \) and \( p_2 \) have the same polynomial order.

### 4.7 Single solution bounds for radii polynomial

For ease of exposition, we will first treat the case of a single solution, i.e. we construct the bounds for the radii polynomials for \( \hat{x} = \hat{x}_s \) solution of \( H_s \) for \( s = 0, 1 \). This may also be interpreted as finding the bounds required for the \( s \)-independent version of Theorem 4.3.1, i.e., the case where \( \hat{x}_0 = \hat{x}_1 \) and \( \hat{q}_0 = \hat{q}_1 \). We note that although the arguments below are self-contained, some additional details can be found in [66].

#### 4.7.1 \( Y \) bound

The \( Y \) bound is an upper bound to

\[ \|T(\hat{x}) - \hat{x}\|_X = \|AH(\hat{x})\|_X. \]
By construction, $\hat{x}$ has $K$ non-zero modes. Furthermore, $F_1$ and $F_2$ are the sum of linear terms and second order convolutions. Thus $F_i(\hat{x}) = \Pi_i(2K) F_i(\hat{x})$ for $i = 1, 2$, and as a consequence $H(\hat{x}) = \Pi(2K) H(\hat{x})$. We know that $A$ is eventually diagonal and Remark 4.4.6 can be applied. Therefore $AH(\hat{x})$ has $2K$ non-zero modes and can be computed with interval arithmetic. Thus, its norm can be computed directly.

4.7.2 $Z$ bound

The $Z$ bound presented in (4.30) can be split into three parts by considering

$$DT(\hat{x} + rb)rc = [(I - AA^\dagger) + A(A^\dagger - DH(\hat{x})) + A(DH(\hat{x}) - DH(\hat{x} + rb))] rc,$$

and computing their bounds separately. We can write

$$Z(r) = Z_0 r + Z_1 r + Z_2(r) r$$

with

$$Z_0 \geq \|I - AA^\dagger\|_{B(X,X)},$$

$$Z_1 \geq \|A(A^\dagger - DH(\hat{x}))\|_{B(X,X)},$$

and

$$Z_2(r) \geq \|A(DH(\hat{x}) - DH(\hat{x} + rb))\|_{B(X,X)},$$

where $I \in B(X,X)$ is the identity operator. In the following, we will split the computation of the $Z$ into the computation of $Z_0$, $Z_1$ and $Z_2(r)$.

4.7.3 $Z_0$ bound

In the case at hand, the $Z_0$ bound is fairly simple, because $A$ and $A^\dagger$ are eventually diagonal operators of center dimension $K$. Thus, applying the triangle inequality we have

$$\|I - AA^\dagger\|_{B(X,X)} = \|I\Pi_{(K)}^{\infty} - A^\dagger A\Pi_{(K)}^{\infty}\|_{B(X,X)} + \|A\Pi_{(K)}^{4} - A^F A^\dagger F\|_{B(X,X)}.$$

We remark that, by definition, the tails of the operator $A$ and $A^\dagger$ are exact inverses of one another. Therefore, the tail of $A^\dagger A^\dagger F$ is the identity, and the first norm is zero.

We are left to compute

$$\|I\Pi_{(K)}^{4} - A^F A^\dagger F\|_{B(X,X)}.$$

The matrix representations of $A^F$ and $A^\dagger F$ are known and finite. Hence we can compute this norm applying Equation (4.33) and the definition of the operator norms in (4.32).
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4.7.4 \( Z_1 \) bound

In the computation of the \( Z_1 \) bound, the subtraction \( A^\dagger - DH(\hat{x}) \) gives an operator where most terms cancel and which acts as a convolution. We can represent it as

\[
B = A^\dagger - DH(\hat{x}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & Kb_{22} & 0 \\ Kb_{31} & Kb_{32} & Kb_{33} \end{pmatrix} - \Pi^{(K)} \begin{pmatrix} 0 & 0 & 0 \\ 0 & Kb_{22} & 0 \\ Kb_{31} & Kb_{32} & Kb_{33} \end{pmatrix} \Pi^{(K)}, \quad (4.62)
\]

with

\[
\begin{align*}
    b_{22} &= 2iC_{\ell_2}^\dagger (\hat{y}), \\
    b_{31} &= i(2\hat{z} * E_\ell \hat{y} + \hat{a}(\hat{z} * \hat{z})), 0, \hat{\lambda}_1 i(\hat{z} * \hat{z}), \\
    b_{32} &= 2\hat{\lambda}_1 iC_{\ell_1}^\dagger(\hat{z})E_\ell, \\
    b_{33} &= 2\hat{\lambda}_1 iC_{\ell_1}^\dagger (E_\ell \hat{y} + \hat{a} \hat{z}),
\end{align*}
\]

where, by an abuse of notation,

\[
Kb_{31} = K \begin{bmatrix} i(2\hat{z} * E_\ell \hat{y} + \hat{a}(\hat{z} * \hat{z})), 0, \hat{\lambda}_1 i(\hat{z} * \hat{z}) \end{bmatrix} = \begin{bmatrix} K_i(2\hat{z} * E_\ell \hat{y} + \hat{a}(\hat{z} * \hat{z})), 0, K\hat{\lambda}_1 i(\hat{z} * \hat{z}) \end{bmatrix}.
\]

**Remark 4.7.1.** In the second block-row of \( B \) the row corresponding to \( k = 0 \), which was “redefined” through \( \tilde{\Pi} \) to correspond to \( G_3^\dagger \) rather than \( (F_1)_0 \), vanishes. Similar considerations hold for the row corresponding to \( j = k = 0 \) in the third block-row of \( B \). Hence the compact notation in (7.5) may be used without including the projection \( \Pi \). In the same way, in the rest of the paper, when the rows corresponding to \( k = 0 \) and \( j = k = 0 \) vanish, we drop the use of the projection \( \Pi \), such as in Equation (4.63) as well as for higher derivatives of \( H \).

We decompose

\[
A(A^\dagger - DH(\hat{x})) = A^F (A^\dagger - DH(\hat{x})) + A^I (A^\dagger - DH(\hat{x})),
\]

and

\[
\|A(A^\dagger - DH(\hat{x}))\| \leq \|A^F (A^\dagger - DH(\hat{x}))\| + \|A^I (A^\dagger - DH(\hat{x}))\|.
\]

The first term has a finite number of non-zero elements, by Theorem 4.4.12. The norm can therefore be computed explicitly. The second term can be
handled by computing explicitly the product $A^T B$ with the help of the two notations introduced in (4.51) and (4.62). Then we have

$$A^T B = \begin{pmatrix} 0 & A_1^T K b_{22} & 0 \\ 0 & A_1^T K b_{32} & A_1^T K b_{33} \end{pmatrix} - \Pi^{(K)} \begin{pmatrix} 0 & 0 & 0 \\ 0 & A_1^T K b_{22} & 0 \\ A_1^T K b_{31} & A_1^T K b_{32} & A_1^T K b_{33} \end{pmatrix},$$

where the second operator vanishes, and it holds that

$$\|A^T B\|_{\infty} \leq \begin{pmatrix} 0 & 0 & 0 \\ 0 & \|A_1^T K\|_{22} b_{22} & 0 \\ \|A_1^T K\|_{33} b_{31} & \|A_1^T K\|_{33} b_{32} & \|A_1^T K\|_{33} b_{33} \end{pmatrix},$$

with the inequality interpreted elementwise. Using Equations (4.35) and (4.37), we obtain

$$\|b_{22}\|_{22} \leq 2\|\hat{y}\|_{L_2^2},$$
$$\|b_{31}\|_{31} \leq \max\{2\hat{z} * E\hat{y} + \hat{a}(\hat{z} * \hat{z})\|_{L_2^1}, 0, \|\hat{\lambda}_1\|\hat{z} * \hat{z}\|_{L_2^1}\},$$
$$\|b_{32}\|_{32} \leq 2\|\hat{\lambda}_1\|\hat{z}\|_{L_2^1},$$
$$\|b_{33}\|_{33} \leq 2\|\hat{\lambda}_1\|E\hat{y} + \hat{a}\hat{z}\|_{L_2^1},$$

and $\|A^T K\|$ for $i = 0, 1$ can be computed with (4.57).

### 4.7.5 $Z_2$ bound

Following the approach already presented in [74], for the computation of $Z_2(r)$ we can apply the mean value theorem. We introduce $R$, an $a priori$ bound on the validation radius $r^*$, and we obtain

$$\|A(DH(\hat{x}) - DH(\hat{x} + rb))\|_{B(X,X)} \leq \sup_{b,c \in B_1(0)} \|AD^2H(\hat{x} + Rb)c\|_{B(X,X)}.$$

The major difference with respect to [74] is that in the case at hand the norm of $D^2H(\hat{x} + Rb)$ is not finite because it includes spatial derivative operators. Therefore, in this case the multiplication with $A$ plays a key role in keeping the $Z_2(r)$ bound finite.

We start by computing $D^2H$ explicitly, using the notation introduced in (4.31). We compute it at a point $x = (\lambda_1, \lambda_2, a, y, z)$ and apply it to $w = (\beta_1, \beta_2, b, r, s) \in X$:

$$D^2H(x)w = \begin{pmatrix} 0 & d_{21}(x, w) & 0 \\ d_{31}(x, w) & d_{22}(x, w) & 0 \\ d_{32}(x, w) & d_{33}(x, w) & 0 \end{pmatrix},$$

(4.63)
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with

\[ d_{21}(x, w) = [0, K^4r, 0], \]
\[ d_{22}(x, w) = K^4\beta_2 - 2iKC_{\ell_2}(r), \]
\[ d_{31}(x, w) = \left[ d_{311}(x, w), K^4(z\beta_1 + \lambda_1s), -iK(\beta_1z + 2\lambda_1z + s) \right], \]
\[ d_{311}(x, w) = K^4(z\beta_2 + \lambda_2s) - K^2s \]
\[ - iK(2E_{\ell y} * s + 2z * E_{\ell r} + bz * z + 2az * s), \]
\[ d_{32}(x, w) = -2iKC_{\ell_1}^1(\lambda_1s + \beta_1z)E_{\ell}, \]
\[ d_{33}(x, w) = K^4(\lambda_2\beta_1 + \lambda_1\beta_2) - K^2\beta_1 \]
\[ - 2iKC_{\ell_1}^2(\beta_1E_{\ell y} + \lambda_1E_{\ell r} + a\beta_1z + \lambda_1(az + as)). \]

We notice that \( D^2H \) is a 2K-diagonal operator since it is the product of diagonal operators and convolution operators. All \( d_{nm}(x, w) \) are of the form

\[ d_{nm}(x, w) = Kd_{nm}^{[1]}(x, w) + K^2d_{nm}^{[2]}(x, w) + K^4d_{nm}^{[4]}(x, w), \]

for \( n, m = 1, \ldots, 3 \).

We use the notation introduced in (4.31) to write

\[ A = \begin{pmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{pmatrix}. \]

With this notation, we can write

\[ (AD^2H(\hat{x} + Rb)c)_{im} = \sum_{n=1,\ldots,3} A_{in}d_{nm}(\hat{x} + Rb, c), \]

and by the triangle inequality we can bound

\[ \|(AD^2H(\hat{x} + Rb)c)_{im}\|_{im} \leq \sum_{n=1,\ldots,3} \|A_{in}d_{nm}(\hat{x} + Rb, c)\|_{im}. \]

With the notation introduced in (4.64) and applying again the triangle inequality, we have

\[ \|A_{in}d_{nm}(\hat{x} + Rb, c)\|_{im} \leq \sum_{p=1,2,4} \|A_{in}K^p\|_{in}\|d_{nm}^{[p]}(\hat{x} + Rb, c)\|_{nm}. \]

Since \( K \) is a diagonal operator, it holds that, by Lemma 4.4.4,

\[ \|A_{ij}K^p\|_{ij} = \max\{\|A_{ij}^F\Pi^{(K)}K^p\|_{ij}, \|A_{ij}^r\Pi^{(K)}K^p\|_{ij}\}, \]

\[ 135 \]
where the first norm can be computed directly and the second can be estimated by applying the bounds from Section 4.6. The norms \( \|d_{nm}^{[p]}(x + Rb, c)\|_{nm} \) can be easily bounded for all \( n \) and \( m \) because they are sums and products of scalars and convolution operators, hence (4.35) and (4.37) and the bound \( \|x + Rb\|_c \leq \|x\|_c + R \) suffice to bound these terms.

As an example we consider \( d_{33}^{[3]}(x, w) \), which we can write as

\[
d_{33}^{[1]}(x, w) = -2iC\mathcal{L}_y^1(\beta_1 E_\ell y + \lambda_1 E_\ell r + a\beta_1 z + \lambda_1 (bz + as)),
\]
\[
d_{33}^{[2]}(x, w) = -\beta_1,
\]
\[
d_{33}^{[4]}(x, w) = \lambda_2 \beta_1 + \lambda_1 \beta_2.
\]

Using \( \|x + Rb\|_c \leq \|x\|_c + R \) and \( \|b\|_c \leq 1 \), we obtain, for any \( \|w\|_X \leq 1 \),

\[
\|d_{33}^{[1]}(x + Rb, w)\|_{33} \leq 2 \left( \|y\|_\ell + |\lambda_1| + 2R + (|a| + R)(\|z\|_{\mathcal{L}_y^1} + R) + (|\lambda_1| + R)(\|\mathcal{L}_y^1 + |a| + 2R) \right),
\]
\[
\|d_{33}^{[2]}(x + Rb, w)\|_{33} \leq 1,
\]
\[
\|d_{33}^{[4]}(x + Rb, w)\|_{33} \leq |\lambda_2| + |\lambda_1| + 2R.
\]

The estimates of the other terms are treated in an analogous manner.

### 4.8 Bounds for segment validation

To compute the bounds introduced in Section 4.3 that include a maximum over \( s \in [0, 1] \), we use the following theorem.

**Theorem 4.8.1.** Let \( f : [0, 1] \to \mathbb{C} \) be \( C^2 \), then

\[
\max_{s \in [0, 1]} |f(s)| \leq \max\{|f(0)|, |f(1)|\} + \frac{1}{8} \max_{x \in [0, 1]} |f''(x)|. \tag{4.65}
\]

For the proof, we refer to [74]. We want to apply this Theorem to the variety of cases at hand. The most general application is the following.

**Lemma 4.8.2.** Let \( i, j \in \{1, 2, 3\} \). For \( s \in [0, 1] \) let \( A(s) : X_i \to X_j \) be a family of bounded linear operators dependent on a parameter \( s \), with matrix representation

\[
A(s) = (A_{k_1,k_2}(s))_{k_1 \in I_1, k_2 \in I_2},
\]

with \( I_1 \) and \( I_2 \) the appropriate set of indices for \( X_i \) and \( X_j \) respectively, that is \( I_1 = \{1, 2, 3\}, I_2 = \mathbb{Z}, I_3 = \ldots \).
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\[ Z^2. \] Then

\[
\max_{s \in [0,1]} \| A(s) \|_{ij} \leq \left( \max \{|A_{k_1,k_2}(0)|, |A_{k_1,k_2}(1)|\} \right)_{k_1 \in I_1, k_2 \in I_2} + \frac{1}{\delta} \left( \sup_{x \in [0,1]} \left| A''_{k_1,k_2}(x) \right| \right)_{k_1 \in I_1, k_2 \in I_2},
\]

provided the elementwise second derivatives \( A''_{k_1,k_2} \) exists for all \( k_1 \in I_1 \) and \( k_2 \in I_2 \) and the second term is finite.

For the proof, we refer to [74]. That proof is carried out on a different space but the proof on \( X \) follows in the same way.

Considering the space \( X \) and the linear operator \( B(s) : X \to X \) for \( s \in [0,1] \), using the notation (4.31), we have

\[
\max_{s \in [0,1]} \| B(s) \|_{B(X,X)} \leq \max_{j=1,\ldots,3} \sum_{i=1,\ldots,3} \max_{s \in [0,1]} \| B_{ij}(s) \|, \quad (4.66)
\]

and we can apply Lemma 4.8.2 to each \( B_{ij} \). Having already introduced the operators \( A_0 \) and \( A_1 \), we introduce a notation for their difference:

\[ A_\Delta = A_1 - A_0. \]

In general, the subscript \( \Delta \) will indicate the difference between the element at hand at \( s = 1 \) and \( s = 0 \).

**Remark 4.8.3.** By construction, \( A_\Delta \) is an eventually diagonal operator of center dimension \( K \), hence \( A_\Delta = A_\Delta^F + A_\Delta^I \). The tail of \( A_\Delta \) is non-zero in general, but applying the same notation as in (4.51), it is computable elementwise by

\[
A_\Delta^I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A_{\Delta_1}^I & 0 \\ 0 & 0 & A_{\Delta_2}^I \end{pmatrix} \Pi_{(K)}^\infty.
\]

Here

\[
A_{\Delta_1}^I = P_1(1)^{-1} - P_1(0)^{-1} = \frac{P_1(0) - P_1(1)}{P_1(0)P_1(1)},
\]

and

\[
A_{\Delta_2}^I = P_2(1)^{-1} - P_2(0)^{-1} = \frac{P_2(0) - P_2(1)}{P_2(0)P_2(1)},
\]

with

\[ P_1(s) \overset{\text{def}}{=} \lambda_{22}K^4 - K^2, \quad \text{for } s = 0,1, \]
and
\[ P_2(s) \overset{\text{def}}{=} iJ + \lambda_{s1}\lambda_{s2}K^4 - \lambda_{s1}K^2, \quad \text{for } s = 0, 1. \]

All inverses, fractions and multiplications are here intended elementwise, because we are working with diagonal operators.

### 4.8.1 Y bound

In the continuation case, we want to compute \( Y \) such that
\[ \max_{s \in [0, 1]} \| A_s H_s(\hat{x}_s) \|_X \leq Y. \]

We denote \( A_s H_s(\hat{x}_s) = (g_1(s), g_2(s), g_3(s)) \in X = \mathbb{R}^3 \times \ell_{v_2}^1 \times \ell_{v_1}^1 \) for \( s \in [0, 1] \).

In a similar way as in Theorem 4.8.2 it holds, for \( i = 1, 2, 3 \), that
\[
\max_{s \in [0, 1]} \| g_i(s) \|_{X_i} \leq \frac{1}{8} \left( \max_{s \in [0, 1]} \left\| g_i''(s) \right\|_{X_i} \right) + \left\| \left( \max \{|(g_i(0))|, |(g_i(1))|\} \right)_{k \in I_i} \right\|_{X_i}.
\]

(4.68)

Applying the approach presented Section 4.7.1 we can compute all nonvanishing components of \( g_i(0) \) and \( g_i(1) \), thus making the second term straightforward to bound explicitly. We are left with having to bound the second order derivative. It holds, by linearity in \( s \) of \( \hat{x}_s \) and \( A_s \), that
\[
g''(s) = D_s D_s (A_s H_s(\hat{x}_s)) = 2A_{\Delta} DH(\hat{x}_s)x_{\Delta} + A_s D_s D_s H(\hat{x}_s)
\]
\[= 2A_{\Delta} DH(\hat{x}_s)x_{\Delta} + A_s DDH(\hat{x}_s)x_{\Delta}x_{\Delta}. \]

By construction, \( x_{\Delta} \in \Pi^{(K)}X \), while \( DH(\hat{x}_s) \) acts on \( x_{\Delta} \) as a second order convolution. Therefore \( DH(\hat{x}_s)x_{\Delta} \in \Pi^{(2K)}X \). By Lemma 4.4.6, thanks to \( A_{\Delta} \) being eventually diagonal of center dimension \( K \), we find that \( A_{\Delta} DH(\hat{x}_s)x_{\Delta} \in \Pi^{(2K)}X \). In the same way we can prove that
\[ A_s DDH(\hat{x}_s)x_{\Delta}x_{\Delta} \in \Pi^{(2K)}X. \]

Thus, we can bound \( \| D_s D_s (A_s H_s(\hat{x}_s)) \|_c \) by a computation where we replace \( \hat{x}_s \) by the interval
\[
x_s = \hat{x}_0 + [0, 1] \cdot x_{\Delta},
\]
interpreted elementwise, and all the computations involving \( \hat{x}_s \) are performed using interval arithmetic. Thus, both terms in (4.68) can be bounded explicitly.
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4.8.2 $Z_0$ bound

Applying the splitting (4.58), the $Z_0$ bound is such that

$$Z_0 \geq \max_{s} \| I - A_s A_s^\dagger \|_{B(X,X)}.$$  

Using the notation (4.31) we apply Lemma 4.8.2 to obtain, for $i, j = 1, 2, 3$,

$$\| (I - A_s A_s^\dagger)_{ij} \|_{ij} \leq$$

$$\left( \max_s \left\| (I - A_0 A_0^\dagger)_{ij} k_1 k_2 \right\|, \left\| (I - A_1 A_1^\dagger)_{ij} k_1 k_2 \right\| \right)_{k_1 \in I_1, k_2 \in I_2} \|_{ij}$$

$$+ \frac{1}{4} \left( \sup_s \left\| (A_\Delta A_\Delta^\dagger)_{ij} k_1 k_2 \right\| \right)_{k_1 \in I_1, k_2 \in I_2}.$$  

The first term is the elementwise maximum over $I - A_0 A_0^\dagger$ and $I - A_1 A_1^\dagger$. We computed these terms and their norm separately in Section 4.7.3. Here we simply need to take the elementwise maximum of their absolute values before taking the norm, but it is otherwise a straightforward extension of the computation in Section 4.7.3. We are left to compute $\| (A_\Delta A_\Delta^\dagger)_{ij} \|_{ij}$, where the maximum over $s$ is dropped, because the argument does not depend on $s$. The product $A_\Delta A_\Delta^\dagger$ is the product of two eventually diagonal operator of center dimension $K$, thus it is again an eventually diagonal operator of center dimension $K$. It then holds that

$$\| I - A_\Delta A_\Delta^\dagger \| \leq \max\{ \| \Pi^{(K)} - A_\Delta A_\Delta^\dagger \|, \| \Pi_{(K)} - A_\Delta A_\Delta^\dagger \| \}$$

The norm of the finite center can be computed directly, hence we will concentrate on the bound of the tail operator.

Let us recall, from the definitions of $A_i$ and $A_i^\dagger$, with $i = 0, 1$, that

$$A_\Delta^I = A_0^I - A_1^I = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \hat{\lambda}_2 \Delta K^4 & 0 & 0 \\ 0 & 0 & (\hat{\lambda}_1 \hat{\lambda}_2) \Delta K^4 & -\hat{\lambda}_1 \Delta K^2 \end{pmatrix} \Pi_{(K)},$$

and $A_\Delta^I$ is defined in (4.67). It follows from Equations (4.70) and (4.67) that

$$A_\Delta^I A_\Delta^I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & B_1 & 0 \\ 0 & 0 & B_2 \end{pmatrix} \Pi_{(K)}.$$
with

\[
B_1 = \frac{(\hat{\lambda}_2 \Delta K^4)(\hat{\lambda}_2 \Delta K^4)}{(\lambda_{21} K^4 - K^2)(\lambda_{20} K^4 - K^2)},
\]

\[
B_2 = \frac{((\hat{\lambda}_1 \hat{\lambda}_2) \Delta K^4 - \hat{\lambda}_1 \Delta K^2)((\hat{\lambda}_1 \hat{\lambda}_2) \Delta K^4 - \hat{\lambda}_1 \Delta K^2)}{(iJ + \lambda_{11} \lambda_{21} K^4 - \lambda_{11} K^2)(iJ + \lambda_{10} \lambda_{20} K^4 - \lambda_{10} K^2)}.
\]

With this format, we could apply directly the results of Section 4.6, but we prefer instead to avoid computing bounds having diagonal terms of order 8. We therefore simplify the computations by considering

\[
\|B_1 \Pi^\top_{(K)}\|_{22} \leq \left\| \frac{\hat{\lambda}_2 \Delta K^4}{\lambda_{21} K^4 - K^2} \Pi^\top_{(K)} \right\|_{22},
\]

\[
\|B_2 \Pi^\top_{(K)}\|_{33} \leq \left\| \frac{(\hat{\lambda}_1 \hat{\lambda}_2) \Delta K^4 - \hat{\lambda}_1 \Delta K^2}{iJ + \lambda_{11} \lambda_{21} K^4 - \lambda_{11} K^2} \Pi^\top_{(K)} \right\|_{33},
\]

In all these four cases, we can apply the approach presented in Section 4.6 to bound the norms of the rational tail operators.

This concludes the computation of the $Z_0$ bound in the continuation setting.

### 4.8.3 $Z_1$ bound

Applying Lemma 4.8.2, and using the notation

\[
A_s(A^I_s - DH_s(\hat{x}_s)) = \begin{pmatrix}
B_{11}(s) & B_{12}(s) & B_{13}(s) \\
B_{21}(s) & B_{22}(s) & B_{23}(s) \\
B_{31}(s) & B_{32}(s) & B_{33}(s)
\end{pmatrix}
\]

we can write

\[
\max_{s \in [0,1]} \sup_{c \in B_s(z)} \|B_{ij}(s)\|_{ij} \leq \left\| \max \{ |(B_{ij}(1))_{k_1 k_2}|, |(B_{ij}(0))_{k_1 k_2}| \} \right\|_{ij} + \\
\frac{1}{8} \left\| \max_{s \in [0,1]} |(B''_{ij}(s))_{k_1 k_2}| \right\|_{ij} \leq Z_1 r,
\]

for $i, j = 1, 2, 3$. The bound at the end points, $s = 0, 1$, follows from the procedure presented in Section 4.7.4 taking the elementwise maxima before taking the norms, as in Section 4.8.2. Therefore we are left to bound the components of

\[
B''(s) = D_s D_s \left( A_s(A^I_s - DH_s(\hat{x}_s)) \right).
\]
4.8. BOUNDS FOR SEGMENT VALIDATION

By linearity of \( A_s \) and \( A_s^\dagger \) in \( s \), it holds that

\[
D_s D_s (A_s (A_s^\dagger - DH_s(\hat{x}_s)))
= 2(A_\Delta D_s (A_s^\dagger - DH_s(\hat{x}_s))) + A_s D_s D_s (A_s^\dagger - DH_s(\hat{x}_s))
= 2A_\Delta A_s^\dagger - 2A_\Delta D^2 H_s(\hat{x}_s)x_\Delta - A_\Delta D^3 H_s(\hat{x}_s)x_\Delta x_\Delta.
\] (4.71)

We already discussed a bound for \( A_\Delta A_s^\dagger \) in Section 4.8.2. For the second term \( A_\Delta D^2 H_s(\hat{x}_s)x_\Delta \), we recall the direct computation of \( D^2 H_s \) in (4.63). Furthermore, we have \( \hat{x}_s, x_\Delta \in \Pi^{(K)} X \). Since \( A_\Delta \) is an eventually diagonal operator, it holds that

\[
A_\Delta D^2 H_s(\hat{x}_s)x_\Delta = A_\Delta^\hat{e}_G \Pi^{(K)}_s D^2 H_s(\hat{x}_s)x_\Delta + A_\Delta^I \Pi^{(K)}_{\hat{e}} D^2 H_s(\hat{x}_s)x_\Delta.
\]

In the first term, we now replace, as in Section 4.8.1, \( \hat{x}_s \) by the interval \( x_s \) defined in (4.69). Since we know that \( D^2 H_s(\hat{x}_s) \) has a finite number of nonvanishing (interval) components, we can explicitly compute all the nonvanishing components of the first term and compute a bound for \( \| A_\Delta^\hat{e}_G \Pi^{(K)}_s D^2 H_s(\hat{x}_s)x_\Delta \|_c \). From now on we replace \( \hat{x}_s \) by \( x_s \) whenever convenient.

For the second term, it holds that

\[
A_\Delta^I \Pi^{(K)}_{\hat{e}} D^2 H_s(\hat{x}_s)x_\Delta = \begin{pmatrix}
0 & 0 & 0 \\
A_\Delta^I d_{22}(\hat{x}_s, x_\Delta) & 0 & 0 \\
0 & A_\Delta^I d_{32}(\hat{x}_s, x_\Delta) & A_\Delta^I d_{33}(\hat{x}_s, x_\Delta)
\end{pmatrix}.
\]

With the notation introduced in (4.67), we can write

\[
A_\Delta^I d_{22}(\hat{x}_s, x_\Delta) = \frac{\hat{\lambda}_{\Delta 2} K^4}{(\hat{\lambda}_{20} K^4 - K^2)(\hat{\lambda}_{21} K^4 - K^2)} \Pi^{(K)}_{\hat{e}} d_{22}(\hat{x}_s, x_\Delta),
\]

\[
A_\Delta^I d_{32}(\hat{x}_s, x_\Delta) = \frac{(\hat{\lambda}_{1} \hat{\lambda}_{2})_{\Delta} K^4 - \hat{\lambda}_{1} K^2}{(iJ + \hat{\lambda}_{10} \hat{\lambda}_{20} K^4 - \lambda_{10} K^2)(iJ + \hat{\lambda}_{11} \hat{\lambda}_{21} K^4 - \lambda_{11} K^2)} \Pi^{(K)}_{\hat{e}} d_{32}(\hat{x}_s, x_\Delta),
\]

\[
A_\Delta^I d_{33}(\hat{x}_s, x_\Delta) = \frac{(\hat{\lambda}_{1} \hat{\lambda}_{2})_{\Delta} K^4 - \hat{\lambda}_{1} K^2}{(iJ + \hat{\lambda}_{10} \hat{\lambda}_{20} K^4 - \lambda_{10} K^2)(iJ + \hat{\lambda}_{11} \hat{\lambda}_{12} K^4 - \lambda_{11} K^2)} \Pi^{(K)}_{\hat{e}} d_{33}(\hat{x}_s, x_\Delta).
\]

We apply a splitting of the fraction similar to the one used in Section 4.8.2
to simplify the computations, thereby retrieving

\[ \| A_{\Delta 1} \delta_{22}(\hat{x}_s, x_{\Delta}) \|_{22} = \left\| \Pi_{\Delta 1}^\infty \frac{d_{22}(\hat{x}_s, x_{\Delta})}{\lambda_{20} K^4 - K^2} \right\|_{22} \left\| \Pi_{\Delta 1}^\infty \frac{\lambda_2 K^4}{\lambda_{21} K^4 - K^2} \right\|_{22}, \]

\[ \| A_{\Delta 2} \delta_{32}(\hat{x}_s, x_{\Delta}) \|_{32} = \left\| \Pi_{\Delta 2}^\infty \frac{d_{32}(\hat{x}_s, x_{\Delta})}{iJ + \tilde{\lambda}_1 \lambda_{20} K^4 - \lambda_{10} K^2} \right\|_{32} \left\| \Pi_{\Delta 2}^\infty \frac{(\tilde{\lambda}_1 \lambda_{21} - \lambda_1) K^4 - \lambda_{10} K^2}{iJ + \tilde{\lambda}_1 \lambda_{21} K^4 - \lambda_{11} K^2} \right\|_{32}, \]

\[ \| A_{\Delta 2} \delta_{33}(\hat{x}_s, x_{\Delta}) \|_{33} = \left\| \Pi_{\Delta 2}^\infty \frac{d_{33}(\hat{x}_s, x_{\Delta})}{iJ + \tilde{\lambda}_1 \lambda_{20} K^4 - \lambda_{10} K^2} \right\|_{33} \left\| \Pi_{\Delta 2}^\infty \frac{(\tilde{\lambda}_1 \lambda_{21} - \lambda_1) K^4 - \lambda_{10} K^2}{iJ + \tilde{\lambda}_1 \lambda_{21} K^4 - \lambda_{11} K^2} \right\|_{33}. \]

The second norm in each bound can be computed by applying the procedure presented in Section 4.6, but the first norms are not yet in that form, because \( d_{ij}(\hat{x}_s, x_{\Delta}) \) are not diagonal operators. Using the expressions for \( d_{ij} \) introduced in Section 4.7.5, we estimate

\[ \left\| \Pi_{\Delta 1}^\infty \frac{d_{22}(\hat{x}_s, x_{\Delta})}{\lambda_{20} K^4 - K^2} \right\|_{22} \leq \left\| \Pi_{\Delta 1}^\infty \frac{K^4}{\lambda_{20} K^4 - K^2} \right\|_{22} |\lambda_{2 \Delta}| + \left\| \Pi_{\Delta 1}^\infty \frac{2K}{\lambda_{20} K^4 - K^2} \right\|_{22} \| y_{\Delta} \|_{L^2}, \]

\[ \left\| \Pi_{\Delta 2}^\infty \frac{d_{32}(\hat{x}_s, x_{\Delta})}{iJ + \tilde{\lambda}_1 \lambda_{20} K^4 - \lambda_{10} K^2} \right\|_{32} \leq \left\| \Pi_{\Delta 2}^\infty \frac{2K}{iJ + \tilde{\lambda}_1 \lambda_{20} K^4 - \lambda_{10} K^2} \right\|_{33} \| \lambda_{1s} z_{\Delta} + \lambda_{1 \Delta} z_{s} \|_{L^2}, \]

\[ \left\| \Pi_{\Delta 2}^\infty \frac{d_{33}(\hat{x}_s, x_{\Delta})}{iJ + \tilde{\lambda}_1 \lambda_{20} K^4 - \lambda_{10} K^2} \right\|_{33} \leq \left\| \Pi_{\Delta 2}^\infty \frac{K^4}{iJ + \tilde{\lambda}_1 \lambda_{20} K^4 - \lambda_{10} K^2} \right\|_{33} |\lambda_{2s} \lambda_{1 \Delta} + \lambda_{1s} \lambda_{2 \Delta}| + \left\| \Pi_{\Delta 2}^\infty \frac{2K}{iJ + \tilde{\lambda}_1 \lambda_{20} K^4 - \lambda_{10} K^2} \right\|_{33} |\lambda_{1 \Delta}| + \left\| \Pi_{\Delta 2}^\infty \frac{2K}{iJ + \tilde{\lambda}_1 \lambda_{20} K^4 - \lambda_{10} K^2} \right\|_{33} \|(E_{\ell} y_{\Delta} + a_{s} z_{s}) \lambda_{1 \Delta} + \lambda_{1s} (a_{s} z_{s} + a_{s} z_{\Delta} + E_{\ell} y_{\Delta}) \|_{L^2}. \]

All these terms can either be bounded by direct interval arithmetic computation (since both \( x_{\Delta} \) and the interval-valued \( \hat{x}_s \) have finitely many non-
vanishing components only) or via the bounds on rational tail operators in
Section 4.6.

We will now bound the final term in (4.71), $A_sD^3H_s(\hat{x}_s)x_\Delta x_\Delta$. This
term first requires the computation of $D^3H_s(\hat{x}_s)x_\Delta x_\Delta$, that is

$$D^3H(\hat{x}_s)x_\Delta x_\Delta = \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \\
d_{31}^3(\hat{x}_s, x_\Delta) & d_{32}^3(\hat{x}_s, x_\Delta) & d_{33}^3(\hat{x}_s, x_\Delta) \end{pmatrix}$$

with

$$d_{31}^3(\hat{x}_s, x_\Delta) = \left[d_{311}^3(\hat{x}_s, x_\Delta), 2K^4z_\Delta\lambda_1, d_{313}^3(\hat{x}_s, x_\Delta)\right],$$

$$d_{311}^3(\hat{x}_s, x_\Delta) = 2K^4z_\Delta\lambda_2 - iK(4E\lambda y_\Delta + 4z_\Delta \lambda_1 + 2z_\Delta a_\Delta),$$

$$d_{313}^3(\hat{x}_s, x_\Delta) = -iK(4\lambda_1 z_\Delta + 2\lambda_1 z_\Delta),$$

$$d_{32}^3(\hat{x}_s, x_\Delta) = -4iK\lambda_1 C\ell_1(\Delta)E_\ell,$$

$$d_{33}^3(\hat{x}_s, x_\Delta) = 2K^4\lambda_1 \lambda_2$$

$$- 4iKC\ell_1(\lambda_1 a_\Delta z_\Delta + \lambda_1 (E\lambda y_\Delta + a_\Delta z_\Delta + a_\Delta z_\Delta)).$$

The norm of $A_sD^3H(\hat{x}_s)x_\Delta x_\Delta$ can be bounded by considering $A_s$ as the
interval $[A_0, A_1]$, interpreted elementwise and using the triangular inequality:

$$\|A_sD^3H(\hat{x}_s)x_\Delta x_\Delta\| \leq \|A_s^T D^3H(\hat{x}_s)x_\Delta x_\Delta\| + \|A_s^F D^3H(\hat{x}_s)x_\Delta x_\Delta\|.$$ 

Here we can bound the finite part with Corollary 4.4.13, since all nonva-
vanishing terms can be enclosed using interval arithmetic. The infinite tail
gives

$$A_s^F D^3H(\hat{x}_s)x_\Delta x_\Delta = \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \\
A_{s2}d_{32}^3 & A_{s1}d_{33}^3 \end{pmatrix},$$

and

$$\|A_{s2}d_{32}^3\|_{32} \leq 4\|A_{s2}K\|_{32}\|\lambda_1 z_\Delta\|_{L^1}.$$ 

$$\|A_{s2}d_{33}^3\|_{33} \leq 2\|A_{s2}K^4\|_{33}\|\lambda_1\|_{L^1} + 4\|A_{s2}K\|_{33}\|4$$

$$\leq \lambda_1 a_\Delta z_\Delta + \lambda_1 (E\lambda y_\Delta + a_\Delta z_\Delta + a_\Delta z_\Delta)\|_{L^1}.$$ 

These terms can be bounded either by direct computation or by the tech-
nique presented in Section 4.6.
4.8.4 $Z_2$ bound

For the $Z_2$ bound in the continuation case, we proceed in the same way as in Section 4.7.5. By applying the mean value theorem we have

$$\max_{s \in [0,1]} \sup_{b,c \in B_1(0)} \left\| A_s (DH_s(\hat{x}_s) - DH_s(\hat{x}_s + rb)c) \right\| \leq \max_{s \in [0,1]} \sup_{b,c \in B_1(0)} \left\| A_s D^2 H_s(\hat{x}_s + Rb)c \right\| r^2.$$

Following the strategy presented in Section 4.7.5 and using the convexity of the norm, we bound

$$\max_{s \in [0,1]} \sup_{b,c \in B_1(0)} \sup_{z \in B_1(0)} \left\| (A_s D^2 H_s(\hat{x}_s + zr)b)c \right\| r^2 \leq \max_{A = \{A_0, A_1\}} \sup_{b,c \in B_1(0)} \sup_{z \in B_1(0)} \left\| (A D^2 H_s(\hat{x}_s + zr)b)c \right\| r^2,$$

and we can apply the same process as in Section 4.7.5 of estimates “monomial by monomial”, with the additional step that

$$\|\hat{x}_s\|_c \leq \max\{\|\hat{x}_0\|_c, \|\hat{x}_1\|_c\},$$

where the component norm $\| \cdot \|_c$ is defined in (4.20).

4.9 Conclusion

We have until now discussed how to validate already computed numerical solutions. Before presenting the results, it is therefore logical to first explain how the first numerical solution $\hat{x} = (\lambda_1, \lambda_2, \hat{a}, \hat{y}, \hat{z})$ is computed. We consider the problem (4.13) projected onto a finite dimensional subset $\Pi^{(K)}_\ell \subset \ell_1^{(K)}$. In this framework, Equation (4.13) becomes a finite dimensional system which we can easily solve numerically for $y$ at any given $\lambda_2 = \gamma$. Then, it is straightforward to compute all the eigenvalues $(\mu_1, \ldots, \mu_{2K+1})$ associated to (4.13) at a given $\gamma$ and $y$. This procedure does not need to be rigorous. We look for a value $\gamma = \lambda_2$ and a corresponding numerical approximate solution $\hat{y} \in \tau^{(K)}_p \ell_1^{(K)}$ of Equation (4.13) such that a pair of eigenvalues $\mu$ and $\tilde{\mu}$ crosses the imaginary axes. Let $v$ be the eigenvector associated to $\mu$. We set

$$\hat{z}_{jk} = \begin{cases} v_k & \text{if } j = 1, |k| \leq K, \\ \bar{v}_{-k} & \text{if } j = -1, |k| \leq K, \\ 0 & \text{otherwise.} \end{cases}$$
We also set $\hat{a} = 0$ as a first approximation, because we assume to be very close to a Hopf bifurcation. Then $\lambda_1$ is set to $|\text{Im}(\mu)|^{-1}$. This construction forms a first numerical approximation for $\hat{x}$. Applying the Newton method allows us to increase the accuracy of our initial approximation. In Figure 4.1, the stationary solution at $\gamma \approx \gamma^*$ close to the Hopf bifurcation is presented. In Figure 4.2, the rescaled time dependent perturbation $z$ is plotted.

Once this approximation has been constructed, the predictor-corrector algorithm, as presented for example in [5] and [74], allows us to numerically follow the solution branch. Then, with the method presented in this article, we rigorously validate each segment and follow the solution branch until the sign of $a$ is proven to change. The arguments in Remarks 4.2.7 and 4.2.8 then imply the existence of a Hopf bifurcation.

Indeed, the MATLAB script `script_HopfKS.m`, which can be found at [68], performs the necessary computation to conclude that we have validated the existence of a Hopf bifurcation in the Kuramoto-Sivashinsky equation from the stationary solution depicted in Figure 4.1 at parameter value $\lambda^*_2 = \gamma^* = 0.29877635811475 + [-r, r]$ with the validation radius $r = 7.8 \cdot 10^{-13}$. For the validation we used 20 temporal modes and 30 spatial modes: $K = (20, 30)$. The weights used for the $L^1_\nu$ and $\ell^1_\nu_2$ space are $\nu = (1.1, 1.1)$. The segment has been followed from $a = -1.0000 \cdot 10^{-8} + [-r, r] < 0$ to $a = 9.0000 \cdot 10^{-8} + [-r, r] > 0$ with $r = 7.8 \cdot 10^{-13}$. 

Figure 4.1: Validated $2\pi$-periodic stationary solution at the Hopf bifurcation point. The validation required 30 spatial modes.
Figure 4.2: Rescaled time and space periodic profile $z$ near the Hopf bifurcation. The validation required 30 spatial and 20 temporal modes.