Nonlinear Waves in Local and Nonlocal Media

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Summary

The global theme of this thesis is the study of pattern forming systems. What we mean by this is perhaps best illustrated using an example from population dynamics. Consider for example a population of unicorns inside an ecosystem without any natural enemies. When the unicorns stay at one place they shall reproduce until a certain population size α (depending, among other things, on birth and dead rates and the carrying capacity of the ecosystem) is reached. This is an example of *nonlinear growth*. Spatial spreading of the unicorns has an opposite effect: when we temporarily ignore the effect of population growth the migration of unicorns will result in the average population density at a given location tending to 0. When we combine these two mechanisms we observe an interesting phenomenon, namely, there is a competition between processes which try to drive the population to α and to 0. In Figure 1 the resulting behaviour is depicted schematically. We observe the emergence of a wave in this system. Since the wave in Figure 1 would not exist without the (nonlinear) growth towards α it is common to call such a wave a *nonlinear wave*.

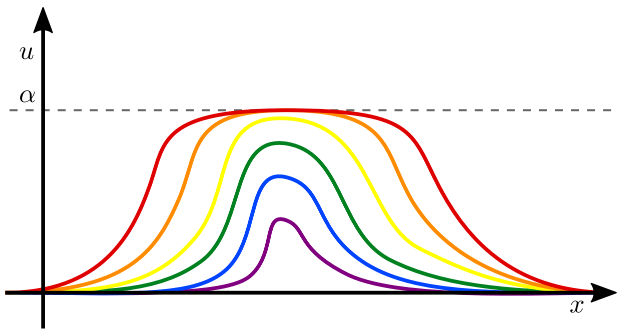

Figure 1: The emergence of a nonlinear wave. From bottom to top along the rainbow the growth of the population u.

The preceding example can be modelled mathematically using so-called *reaction-diffusion equations*. Here the word "reaction" refers to the (nonlinear) growth, whereas "diffusion" is a model for spatial spreading. Note that the nonlinear wave in Figure 1 consists of two interfaces between α and 0, one propagating towards the left and one towards the right. Such a moving interface between two states is called a *travelling wave*. Depicted on the front cover of this thesis is a schematic representation of a travelling wave.

It turns out that the existence of travelling waves is a generic property of reaction-diffusion equations. This is the subject of Chapter 2, where we study models of travelling waves in a general class of reaction-diffusion equations. The class of models which we consider has the special property that associated to them is a certain *energy landscape*. The contour lines of such an energy landscape are depicted on the back cover of this thesis. The special property of these landscapes is that solutions to the models can only descend along the landscapes. We can therefore draw an analogy between finding travelling waves and skiing along such a landscape. This has major consequences, since for example we cannot ski around in a circle and therefore there are no periodic solutions to the model for travelling waves. Sometimes we will not be able to move any further since we have reached a mountain pass or valley. Mathematicians call such location the *critical points* of the landscape. These critical points correspond precisely to the states between which travelling waves may be found.

Now imagine that our energy landscape is located next to an ocean and the dykes break. Consequently, the landscape will slowly get flooded. In this situation the critical points of the landscape also play an important role. Any time the water level reaches the height of a critical point the topography of the landscape will change drastically. For example, what used to be a peninsula can suddenly turn into an island after an increase of the water level. Another example are mountain peaks which may disappear entirely underneath the water. It turns out that solely the manner in which the topography changes already contains information about the travelling waves. For this reason the theory developed in Chapter 2 is called a *topological method*.

The only words from the title of this thesis which we have not yet explained are *local* and *nonlocal media*. Here "media" refers to the space of possible spatial configurations, in the preceding example this was the ecosystem. The reaction-diffusion equations which we described above are examples of models in local media. Here the underlying assumption is that the nonlinear growth and spatial spreading take place at comparable timescales. This is certainly not always a reasonable assumption. One may think for example about birds which may travel long distances over short amounts of time, plants which can compete over long distances for nutrients using an extensive root system, or neurons which can communicate with one another over long distances. Situations where the spatial spreading occurs much faster than the nonlinear growth are often better modelled in nonlocal media. In nonlocal media the spatial spreading can take place instantaneously. Such models are the topic of the last two chapters of this thesis.

In Chapter 3 we describe a class of models for waves in nonlocal media for which we can again assign energy landscapes to the waves. The existence of these energy landscapes turns out to be related to symmetries in the models. Compared to reaction-diffusion equations this is much harder to prove since standard calculus results are not applicable. Furthermore we show that certain waves with a small amplitude can in essence be described using classical mechanics.

Finally, in Chapter 4 we combine the ideas from both Chapter 2 and Chapter 3. There we show that also in models for travelling waves in nonlocal media the topography of the energy landscape contains information about the travelling waves. With this we are able to establish the existence of waves in nonlocal media, which were out of reach using previously known methods.